MATH10222: SOLUTIONS TO EXAMPLE SHEET[1] III

1. **Existence and uniqueness for linear second-order ODEs**

(a) To assess the existence and uniqness of this (linear!) ODE we rewrite

$$x^2\,y'' - 2\,x\,y' + 2\,y = 0$$

in its standard form $y'' + p(x)\,y' + q(x)\,y = r(x)$,

$$y'' - 2\,\frac{1}{x}\,y' + 2\,\frac{1}{x^2}\,y = 0.$$

Because the equation is linear, the existence and uniqueness of its solution is determined by the coefficients $p(x) = -2/x$, $q(x) = 2/x^2$ and $r(x) = 0$. These are continuous in the interval $I = \{x \mid 0 < x\}$ that contains the point $x = 1$ at which the initial condition is specified. Hence a unique solution for the initial value problem exists for all $x \in I$. Some of the coefficients are singular at $x = 0$, so it is not certain that it will be possible to extend the solution across this point. Note that we already had to exclude the case $x = 0$ when transforming the ODE into its standard form, as its highest derivative vanishes at this point – this is always "a sign of trouble". However, recall that the existence and uniqueness theorem does **not** say that the solution will **not** exist for $x \le 0$ – the theorem simply remains quiet about this. [In fact, you can check by inspection that the solution to the IVP is $y(x) = x^2$ which exists for all values of x.]

(b) The ODE

$$\ddot{x} - 2\,\frac{1}{t}\,\dot{x} + 2\,\frac{1}{t^2}\,x = 0$$

is already in its standard form which is the same as that in the previous example. Here the initial conditions are applied at $t = -1$ so a unique solution for the initial value problem is guaranteed to exist in the interval $I = \{t \mid t < 0\}$.

(c) The ODE

$$\ddot{y} + \Omega^2\,y = 0$$

is already in its standard form. Its coefficients are constants and are therefore continuous for $t \in \mathbb{R}$ so a unique solution for the initial value problem exists $t \in I = \mathbb{R}$.

Given that the ODE and the initial conditions are homogeneous, the unique solution of the initial value problem is obviously $y \equiv 0$.

(d) Clearly, the function $y \equiv 0$ is also a solution of the boundary value problem

$$\ddot{y} + \Omega^2\,y = 0,$$

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subject to

$$y(t=0)=0 \quad \text{and} \quad y(t=1)=0.$$

However, the solution $y \equiv 0$ is not necessarily unique as, for certain values of Ω, additional solutions exist: If $\Omega = \Omega_j = j\pi$ (where $j = 0, 1, ...$) the functions $y_j(t) = A \sin(\Omega t)$, where A is an arbitrary constant, also satisfy the ODE and the boundary conditions. These functions are known as eigenfunctions, and the corresponding values of Ω as eigenvalues. Note the similarity with eigenvectors/eigenvalues in linear algebra. Eigenvalue problems for ODEs will be explored in more detail in the second year.

2. **Linear and nonlinear second-order ODEs**

(a)

$$t^2\,\ddot{y} - 2\,t\,\dot{y} + 2\,y = 0$$

- Show that $y_1(t) = t$ is a solution:

$$\dot{y}_1 = 1, \qquad \ddot{y}_1 = 0.$$

Substitute into the ODE:

$$t^2 \times 0 - 2\,t \times 1 + 2 \times t = -2\,t + 2\,t = 0 \qquad \text{Q.E.D.}$$

- Show that $y_2(t) = t^2$ is a solution:

$$\dot{y}_1 = 2\,t, \qquad \ddot{y}_1 = 2.$$

Substitute into the ODE:

$$t^2 \times 2 - 2\,t \times 2\,t + 2 \times t^2 = 2\,t^2 - 4\,t^2 + 2\,t^2 = 0 \qquad \text{Q.E.D.}$$

- $\Longrightarrow$ $y_1(t)$ and $y_2(t)$ are nonzero solutions of the linear ODE. Are they linearly independent? Check by examining if

$$A\,y_1(t) + B\,y_2(t) = 0 \;\; \forall t$$

requires $A \equiv B \equiv 0$.

$$A\,t + B\,t^2 = 0 \;\; \forall t$$

Evaluating this at $t = 1 \Longrightarrow A+B = 0$; evaluating at $t = 2 \Longrightarrow 2A+4B = 0$. Inserting $A = -B$ from the first constraint into the second one gives $B = A = 0$, so the two solutions are linearly independent.

- Since the ODE is homogeneous and linear, and since $y_1(t)$ and $y_2(t)$ are two nonzero, linearly independent solutions, the general solution of the ODE is given by

$$y(t) = A\,t + B\,t^2.$$

Note: The ODE considered here is the same (apart from a trivial renaming of the dependent and independent variables) as the one considered in question 1a where we had established the existence and uniqueness of a solution for $t \in I$ where $I = \{t \mid t > 0\}$. We stressed that the theorem that guaranteed the existence and uniqueness of the solution for strictly positive values of t did *not* imply the non-existence of the solution for other values of t. Indeed the general solution obtained here indicates that the solution is well-behaved for all values of t. However, what would happen if we tried to apply initial conditions at the "singular point" $t = 0$? Recall that the two initial conditions (in the form of constraints on the solution and its first derivative) determine the constants of integration A and B in the general solution. At $t = 0$ (where the existence and uniqueness theorem remains "silent") the general solution is identically equal to zero for *any* values of A and B, so in general we cannot satisfy the initial conditions. This implies the non-existence of the solution for "most" initial conditions (i.e. for all initial conditions other than those that include $y(t = 0) = 0$) and non-uniqueness whenever the solution exists.

(b)

$$y\,\ddot{y} - (\dot{y})^2 = 0$$

- Show that $y_1(t) = e^t$ is a solution:

$$\dot{y}_1 = e^t, \qquad \ddot{y}_1 = e^t.$$

Substitute into the ODE:

$$e^t\,e^t - e^{2t} = 0 \qquad \text{Q.E.D.}$$

- Show that $y_2(t) = e^{2\,t}$ is a solution:

$$\dot{y}_1 = 2\,e^{2\,t}, \qquad \ddot{y}_1 = 4\,e^{2\,t}.$$

Substitute into the ODE:

$$e^{2\,t}\,4\,e^{2\,t} - (2\,e^{2\,t})^2 = 4\,e^{4\,t} - 4\,e^{4\,t} = 0. \qquad \text{Q.E.D.}$$

- ...but $y = A\,e^t + B\,e^{2\,t}$ is **not** a solution:

$$\begin{aligned}
&(A\,e^t + B\,e^{2\,t})\,(A\,e^t + 4\,B\,e^{2\,t}) - (A\,e^t + 2\,B\,e^{2\,t})^2 \\
&= A^2\,e^{2\,t} + 5\,A\,B\,e^{3\,t} + 4\,B^2\,e^{4\,t} - A^2\,e^{2\,t} - 4\,A\,B\,e^{3\,t} - 4\,B^2\,e^{4\,t} \\
&= A\,B\,e^{3\,t} \neq 0
\end{aligned}$$

$A\,y_1(t) + B\,y_2(t)$ is **not** a solution because the ODE is **nonlinear**.

(c) We know that

$$y_1'' + p(x)\,y_1' + q(x)\,y_1 = 0,$$

and

$$y_2'' + p(x)\,y_2' + q(x)\,y_2 = 0.$$

Substitute $A\,y_1(x) + B\,y_2(x)$ into the ODE and re-arrange, using the linearity of the differentiation:

$$\big(A\,y_1 + B\,y_2\big)'' + p(x)\big(A\,y_1 + B\,y_2\big)' + q(x)\big(A\,y_1 + B\,y_2\big) = 0,$$

$$\big(A\,y_1'' + B\,y_2''\big) + p(x)\big(A\,y_1' + B\,y_2'\big) + q(x)\big(A\,y_1 + B\,y_2\big) = 0,$$
$$A\,\underbrace{\big(y_1'' + p(x)\,y_1' + q(x)\,y_1\big)}_{=0} + B\,\underbrace{\big(y_2'' + p(x)\,y_2' + q(x)\,y_2\big)}_{=0} = 0,$$
$$0 = 0 \qquad \text{Q.E.D.}$$

3. **Homogeneous linear ODEs with constant coefficients**

(a) $\ddot{y} - 5\,\dot{y} + 4\,y = 0$

- Characteristic equation $\quad \lambda^2 - 5\,\lambda + 4 = 0 \quad \Longrightarrow \quad \lambda = 1, 4.$
- The general solution is
$$y = C\,e^{t} + D\,e^{4\,t}.$$
- Initial conditions: $\quad y(0) = 0, \quad \dot{y}(0) = 1$
$$\dot{y} = C\,e^{t} + 4\,D\,e^{4\,t}$$

so
$$\begin{cases} y(0) = C + D = 0 \\ \dot{y}(0) = 4\,D + C = 1 \end{cases} \Longrightarrow \begin{cases} D = -C \\ -3\,C = 1 \end{cases} \Longrightarrow \begin{cases} C = -\frac{1}{3} \\ D = \frac{1}{3} \end{cases}$$

- The required solution is
$$y = \frac{1}{3}\,(e^{4\,t} - e^{t}).$$

(b) $\ddot{y} + 4\,y = 0$

- Characteristic equation $\quad \lambda^2 + 4 = 0 \quad \Longrightarrow \quad \lambda = \pm 2\,i.$
- The general solution is:
  - Complex form: $y = C\,e^{i2t} + D\,e^{-i2t}$.
  - Real form: $y = A\,\cos 2t + B\,\sin 2t$.
- Initial conditions: $y(0) = 1, \quad \dot{y}(0) = 0$
$$\dot{y} = -2\,A\,\sin 2t + 2\,B\,\cos 2t$$

so
$$\begin{cases} y(0) = A = 1 \\ \dot{y}(0) = 2\,B = 0 \end{cases}$$

- The required solution is
$$y = \cos 2t.$$

(c) $\ddot{y} - y = 0$

- Characteristic equation $\quad \lambda^2 - 1 = 0 \quad \Longrightarrow \quad \lambda = \pm 1.$
- The general solution is
$$y = C\,e^{t} + D\,e^{-t}.$$
- Initial conditions: $y(0) = 1, \quad \dot{y}(0) = 0$
$$\dot{y} = C\,e^{t} - D\,e^{-t}$$

so
$$\begin{cases} y(0) = C + D = 1 \\ \dot{y}(0) = C - D = 0 \end{cases} \Longrightarrow \begin{cases} C = \frac{1}{2} \\ D = \frac{1}{2} \end{cases}$$

- The required solution is

$$y = \frac{1}{2}(e^t + e^{-t}) = \cosh t.$$

(d) $\ddot{y} + 4\,\dot{y} + 4\,y = 0$

- Characteristic equation $\lambda^2 + 4\,\lambda + 4 = 0 \quad \Longrightarrow \quad \lambda_{1,2} = -2$ (repeated root!)
- The general solution is

$$y = (C + D\,t)\,e^{-2\,t}.$$

- Initial conditions: $y(0) = 1, \quad \dot{y}(0) = -2$

$$\dot{y} = D\,e^{-2\,t} - 2\,(C + D\,t)\,e^{-2\,t}$$

so

$$\begin{cases} y(0) = C = 1 \\ \dot{y}(0) = D - 2\,C = -2 \end{cases} \Longrightarrow \begin{cases} C = 1 \\ D = 0 \end{cases}$$

- The required solution is

$$y = e^{-2\,t}.$$

(e) $\ddot{y} - 2\,\dot{y} + 3\,y = 0$

- Characteristic equation $\lambda^2 - 2\,\lambda + 3 = 0 \quad \Longrightarrow \quad \lambda = 1 \pm i\,\sqrt{2}.$

- The general solution is
  - Complex form: $y = C\,e^{(1+i\,\sqrt{2})\,t} + D\,e^{(1-i\,\sqrt{2})\,t}$,
  - Real form: $y = e^t\,\big(A\,\cos(\sqrt{2}t) + B\,\sin(\sqrt{2}t)\big)$.
- Initial conditions: $y(0) = 0, \quad \dot{y}(0) = \sqrt{2}$

$$\begin{aligned} \dot{y} &= e^t\,\big(A\,\cos(\sqrt{2}t) + B\,\sin(\sqrt{2}t)\big) + e^t\,\big(-\sqrt{2}A\,\sin(\sqrt{2}t) + \sqrt{2}B\,\cos(\sqrt{2}t)\big) \\ &= e^t\,\big((A + \sqrt{2}B)\,\cos(\sqrt{2}t) + (B - \sqrt{2}A)\,\sin(\sqrt{2}t)\big) \end{aligned}$$

so

$$\begin{cases} y(0) = A = 0 \\ \dot{y}(0) = (A + \sqrt{2}B) = \sqrt{2} \end{cases} \Longrightarrow \begin{cases} A = 0 \\ B = 1 \end{cases}$$

- The required solution is

$$y = e^t\,\sin(\sqrt{2}t).$$

(f) $\ddot{y} = 0$

- Characteristic equation: $\lambda^2 = 0 \quad \Longrightarrow \quad \lambda_{1,2} = 0$ (repeated root!)

- The general solution is

$$y = (C\, + D\,t)\,e^{0\times t} = C + D\,t.$$

[We could, of course, have obtained this solution directly by integrating the ODE twice.]

- Initial conditions: $y(0) = 1, \quad \dot{y}(0) = -2$

$$\dot{y} = D$$

so

$$\left\{ \begin{array}{l} y(0) = C = 1 \\ \dot{y}(0) = D = -2 \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} C = 1 \\ D = -2 \end{array} \right.$$

- The required solution is

$$y = 1 - 2\,t.$$

4. **The real form of the fundamental solutions in the case of complex conjugate roots of the characteristic polynomial**
We know that the general solution

$$y(x) = e^{\mu x} \left(\widehat{A}\, e^{i\omega x} + \widehat{B}\, e^{-i\omega x} \right)$$

is real.

$$\begin{aligned} \widehat{A}\, e^{i\omega x} + \widehat{B}\, e^{-i\omega x} &= (\alpha + i\beta)\, e^{i\omega x} + (\gamma + i\delta)\, e^{-i\omega x} \\ &= (\alpha + i\beta) \big(\cos(\omega x) + i \sin(\omega x) \big) + (\gamma + i\delta) \big(\cos(\omega x) - i \sin(\omega x) \big) \\ &= \big(\alpha \cos(\omega x) - \beta \sin(\omega x) + \gamma \cos(\omega x) + \delta \sin(\omega x) \big) + \\ & \quad\; i \, \big(\beta \cos(\omega x) + \alpha \sin(\omega x) + \delta \cos(\omega x) - \gamma \sin(\omega x) \big) \end{aligned}$$

The imaginary part of this expression only vanishes if

$$\delta = -\beta \quad \text{and} \quad \gamma = \alpha.$$

Use this to re-write the real part:

$$\begin{aligned} \widehat{A}\, e^{i\omega x} + \widehat{B}\, e^{-i\omega x} &= (\alpha + \gamma) \cos(\omega x) - (\beta - \delta) \sin(\omega x) \\ &= 2\alpha \cos(\omega x) - 2\beta \sin(\omega x), \end{aligned}$$

so

$$y(x) = e^{\mu x} \left(A \cos(\omega x) + B \sin(\omega x) \right)$$

(setting $A = 2\alpha$ and $B = -2\beta$), as claimed in the lecture.

5. **The form of the solution for repeated roots – "reduction of order"**
Try $y_2(t) = g(t)\, y_1(t) = g(t)\, e^{-k\,t}$ as an ansatz for the second solution. The derivatives are given by

$$\begin{aligned} \dot{y}_2 &= \dot{g}(t)\, e^{-kt} - k\, g(t)\, e^{-k\,t}, \\ \ddot{y}_2 &= e^{-k\,t} \left(\ddot{g}(t) - 2\,k\, \dot{g}(t) + k^2\, g(t) \right). \end{aligned}$$

Substituting this into the ODE and cancelling the common factor $e^{-k\,t}$ yields

$$\ddot{g}(t) - 2\,k\, \dot{g}(t) + k^2\, g(t) + 2\,k \left(\dot{g}(t) - k\, g(t) \right) + k^2\, g(t) = 0.$$

Most terms cancel leaving

$$\ddot{g}(t) = 0,$$

so $g(t) = A + B\,t$, where A and B are arbitrary constants. Hence, a second solution of the ODE is given by

$$y_2(t) = (A + B\,t)\,e^{-k\,t}$$

for any value of the constants A and B.

Note, however, that we want $y_1(t)$ and $y_2(t)$ to be fundamental solutions of the ODE. This requires the two functions to be nonzero and linearly independent, ruling out certain combinations of A and B. For instance, $A = B = 0$ is not a sensible choice as it produces the trivial solution, $y_2 \equiv 0$. Equally, if we set $B = 0$, $y_1(t)$ and $y_2(t)$ are simply multiples of each other and therefore linearly dependent. The easiest way to construct a second fundamental solution is to do what we did in the lecture, namely set $A = 0$ and $B = 1$, yielding the set of fundamental solutions

$$\left\{e^{-k\,t},\ t\ e^{-k\,t}\right\},$$

implying that any solution of the ODE can be written as

$$y(t) = C\,e^{-k\,t} + D\,t\ e^{-k\,t} \tag{1}$$

for arbitrary constants C and D.

However, other choices are possible too. For instance,

$$\left\{e^{-k\,t},\ (1+t)\ e^{-k\,t}\right\}$$

is another set of fundamental solutions, illustrating the statement made in the lecture that the set of fundamental solutions is not unique.

To show that the two sets of fundamental solutions are equivalent, we use the second set to write the general solution $y(t)$ as

$$y(t) = E\,e^{-k\,t} + F\,(1+t)\ e^{-k\,t} \tag{2}$$

for arbitrary constants E and F. Are the two representations (1) and (2) equivalent? Check by equating them:

$$C\,e^{-k\,t} + D\,t\ e^{-k\,t} = E\,e^{-k\,t} + F\,(1+t)\ e^{-k\,t}$$

$$(C - E - F)\,e^{-k\,t} + (D - F)\,t\ e^{-k\,t} = 0$$

Since $e^{-k\,t}$ and $t\,e^{-k\,t}$ are linearly independent this requires $D = F$ and $E = C - F = C - D$. Hence any solution $y(t)$ that is represented by (1) with constants C and D can be represented by (2) with coefficients $F = D$ and $E = C - D$.