Where have we (you!) seen $x = x_P + x_H$ before?

Recall:

The *general* solution of the inhomogeneous ODE

$$y'' + p(x)\, y' + q(x)\, y = r(x) \qquad \text{(I)}$$

can be written as

$$y(x) = y_p(x) + \alpha\, y_1(x) + \beta\, y_2(x),$$

where:

- α and β are arbitrary constants.
- $y_p(x)$ is any particular solution of the inhomogeneous ODE.
- $y_1(x)$ and $y_2(x)$ are fundamental solutions of the corresponding homogeneous ODE.

Compare this to the solution of the system of linear (algebraic) equations:

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

where $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{b}$ a given vector of size n.

The general solution $\mathbf{x}$ (another vector of size n) is given by

$$\mathbf{x} = \mathbf{x}_P + \mathbf{x}_H$$

where

- $\mathbf{x}_P$ is a(ny) particular solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$
- $\mathbf{x}_H$ is the *general* solution of the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$.

Example

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 3 & -3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Note that the matrix is singular, so $\mathbf{Ax} = \mathbf{0}$ has non-trivial solutions!

- Transform into "triangular" form

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

  showing that the RHS is consistent. We're left with one equation for three unknowns.

- Set $x_2 = \alpha$ and $x_3 = \beta$, where α and β are arbitrary constants.

- The general solution is: $x_1 = 1 + \alpha$ and, of course, $x_2 = \alpha$ and $x_3 = \beta$.

- Rewrite in vector form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{x}_P} + \underbrace{\alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{x}_H}$$

- Note that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 3.1415 \end{pmatrix}}_{\mathbf{x}'_P} + \underbrace{\alpha' \begin{pmatrix} -42.2 \\ -42.2 \\ 1145.2 \end{pmatrix} + \beta' \begin{pmatrix} 523.2 \\ 523.2 \\ 13.423 \end{pmatrix}}_{\mathbf{x}'_H}$$

  is another (not so pretty) representation of the general solution.

The key features of both solutions are:

- $\mathbf{x}_P$ and $\mathbf{x}'_P$ solve the inhomogeneous equation.
- $\mathbf{x}_H$ and $\mathbf{x}'_H$ "span the null space" of $\mathbf{A}$, i.e. they
    1. satisfy $\mathbf{Ax} = \mathbf{0}$,
    2. are nonzero,
    3. are linearly independent.

"Off the record comment":

In linear algebra it's "easier" to overlook the additional solutions represented by $\mathbf{x}_H$. In an ODE context, the fact that BCs [or ICs] have to be satisfied too, tends to provide an instant "reminder" that just having *a* particular solution of the ODE is not enough to solve the entire IVP/BVP.