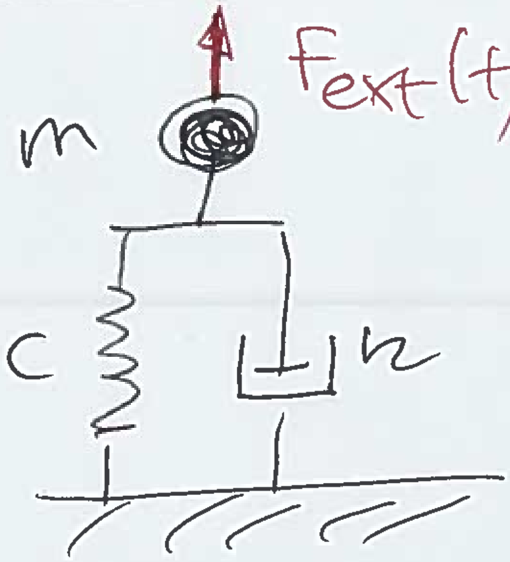


$F_{ext}(t) = \hat{F}_{ext}\cos(\Omega t)$

$$m\ddot{x} + k\dot{x} + cx = \underbrace{\hat{F}_{ext}\cos(\Omega t)}_{Re(\hat{F}_{ext}e^{i\Omega t})}$$

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = \hat{f}\cos(\Omega t)$$

$$\delta = \frac{k}{2m} \; ; \quad \omega^2 = \frac{c}{m} \; ; \quad \hat{f} = \frac{\hat{F}_{ext}}{m}$$

$$x(t) = e^{-\delta t}\left[A\cos\left(\sqrt{\omega^2-\delta^2}\,t\right) + B\sin\left(\sqrt{\omega^2-\delta^2}\,t\right)\right] + Re\left(\bar{X} e^{i\Omega t}\right)$$

$$|X| = \frac{\hat{f}}{\sqrt{(\omega^2-\Omega^2)^2+(2\delta\Omega)^2}}$$

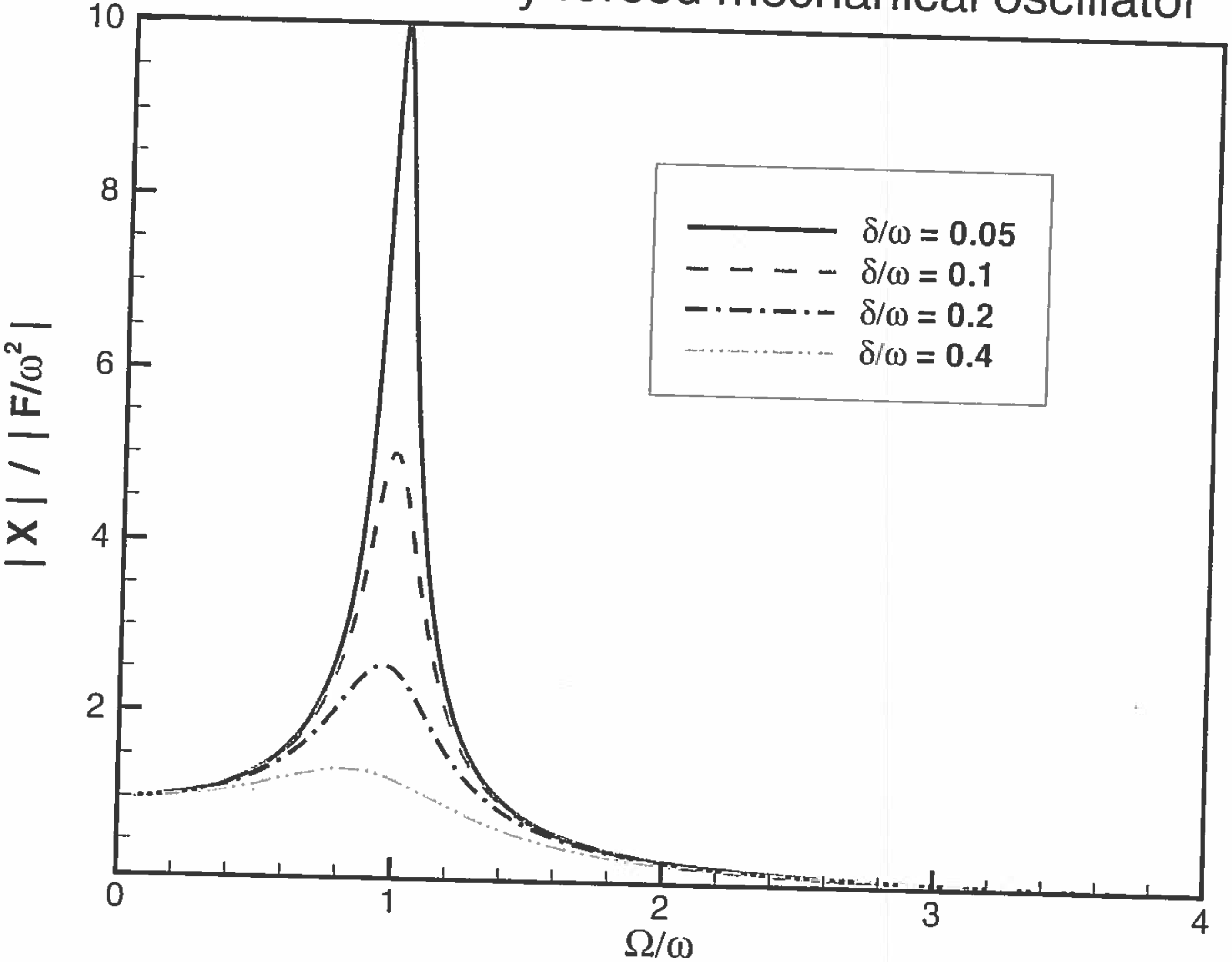
$$\frac{|X|}{\left(\hat{f}/\omega^2\right)} = \frac{1}{\sqrt{\left(1-\left(\frac{\Omega}{\omega}\right)^2\right)^2+\left(2\left(\frac{\delta}{\omega}\right)\left(\frac{\Omega}{\omega}\right)\right)^2}}$$

ratio of frequencies

ratio of damping to spring stiffness

$$\frac{\hat{f}}{\omega^2} = \frac{\hat{F}_{ext}\,\cancel{m}}{\cancel{m}\;c} =$$ static extension of spring in response to $\hat{F}_{ext}$

Normalised amplitude of the oscillation of the harmonically forced mechanical oscillator



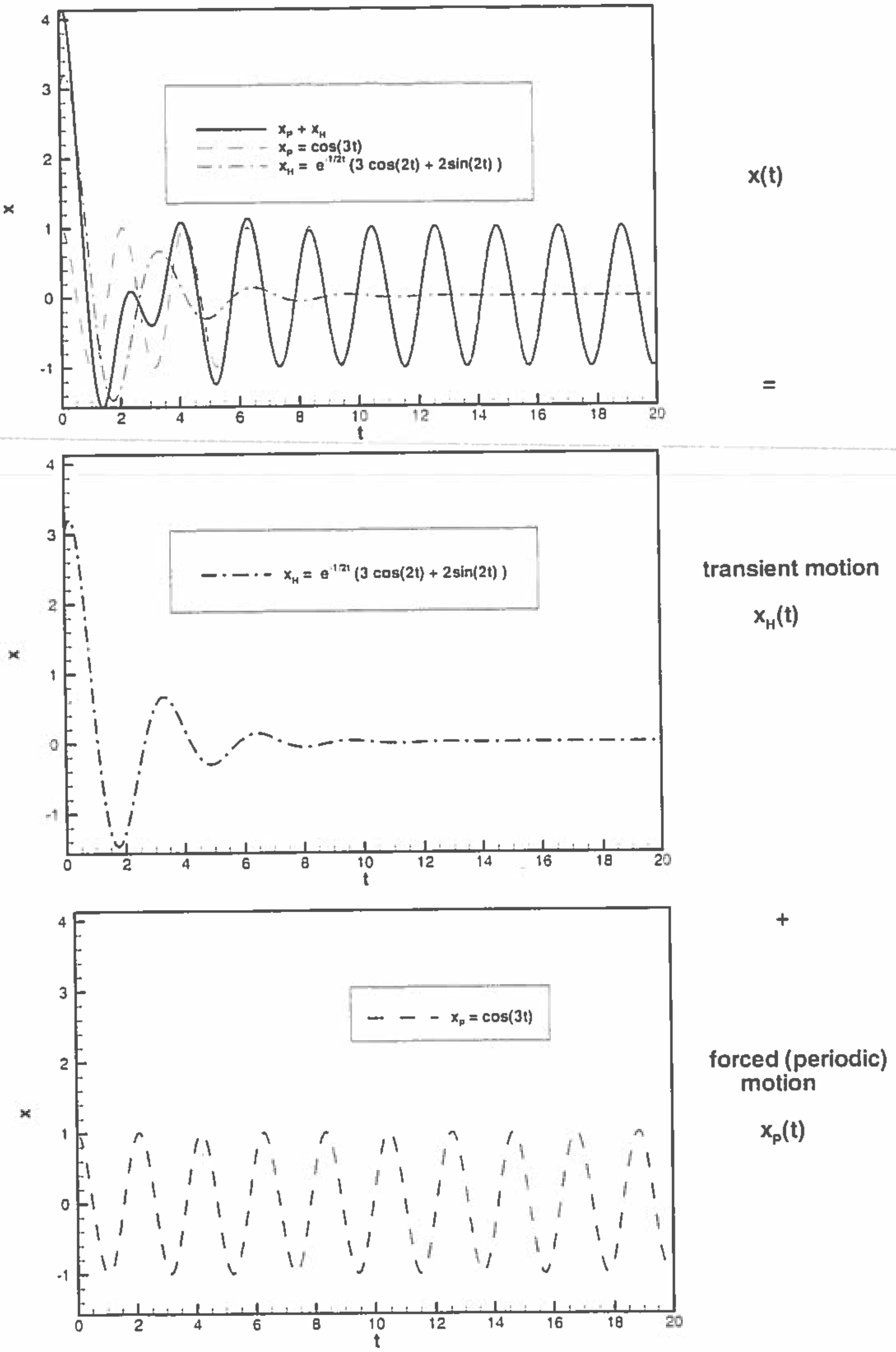


Figure 5: The displacement of a harmonically-forced, damped mechanical oscillator comprises the periodic (forced) solution $x_P(t)$ and the transient solution $x_H(t)$.

Resonance

No damping: $\delta = 0$

$$\ddot{x} + \omega^2 x = \hat{f}\cos(\Omega t) = \mathrm{Re}(\hat{f} e^{i\Omega t})$$

As above unless $\Omega = \omega$

In that case $\cancel{f}\, e^{i\Omega t}$ is a soln. of the homog. ODE.

In that case:

$$x_p = C' t e^{i\Omega t}$$

then take real part

$$\ddot{x}_p = C' e^{i\Omega t}(2i\Omega - \Omega^2 t)$$

into ODE:

$$C' \cancel{e^{i\Omega t}}\Big[\underbrace{2i\Omega - \cancel{\Omega^2 t}}_{\ddot{x}} + \underbrace{\cancel{\Omega^2 t}}_{\omega^2 x}\Big] = \hat{f}\cancel{e^{i\Omega t}}$$

$$C' = \frac{\hat{f}}{2i\Omega} = -i\frac{\hat{f}}{2\Omega}$$

So general soln:

$$x(t) = A\cos(\Omega t) + B\sin(\Omega t) + \mathrm{Re}\left(-i\frac{\hat{f}}{2\Omega}t\underbrace{e^{i\Omega t}}_{\cos(\Omega t)+i\sin(\Omega t)}\right)$$

$$\boxed{x(t) = A\cos(\Omega t) + B\sin(\Omega t) + \underbrace{\frac{\hat{f}}{2\Omega}t\sin(\Omega t)}_{\to\infty \text{ as } t\to\infty}}$$

Basic ideas of perturbation methods: "Exploiting small parameters" and "Scaling"

Observation 1:

- ODEs (and hence their solutions!) typically contain some parameters, e.g.

$$m\ddot{x} + k\dot{x} + cx = F\cos(\Omega t)$$

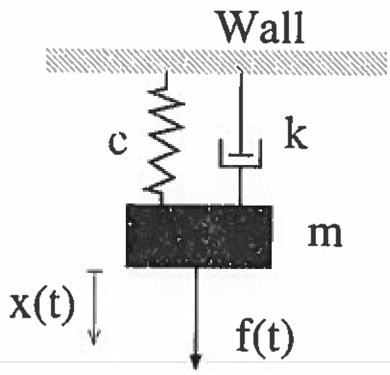
so

$$x = x(t) = x(t; m, k, c, \Omega).$$

- Often some of the problem's parameters are "small". How can we exploit this?
- Example:
  - Assume that we (only) know the solution of the above ODE for $k = 0$ (no damping).
  - What is the solution for "small" k?

Observation 2:

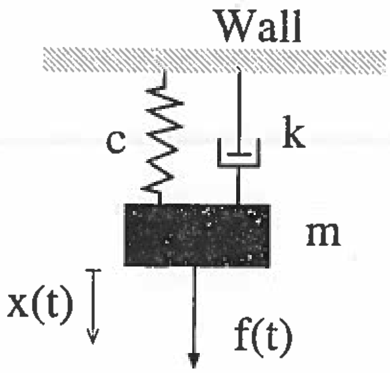
- ODEs that model physical phenomena typically express balances (of forces, energies, currents, ...).
- Here's an example of a balance of forces:



$$\underbrace{m\ddot{x}}_{\text{inertial forces}} + \underbrace{k\dot{x}}_{\text{damping forces}} + \underbrace{cx}_{\text{spring forces}} = \underbrace{F\cos(\Omega t)}_{\text{applied external force}}$$

- In general, all terms in the ODE will make a significant contribution to the overall "balance".
- However, there *may* be regimes in which the balance of terms is dominated by a balance between just a few (ideally two) terms, while the other terms only provide "negligible" contributions.
- The simplified equations (obtained by neglecting the small terms) are often much easier to solve than the full equations.
- We may [should!] then be interested in finding the effect that the "small" perturbations have on the solution.
- A seemingly trivial observation: You will need *at least* two terms to balance!

Example:



$$m\ddot{x} + k\dot{x} + cx = F\ \cos(\Omega t)$$

- We established earlier that

$$x(t) = x_P(t) + x_H(t)$$

where $x_H(t) \to 0$ very rapidly.

- Following the decay of the initial transients [described by $x_H(t)$] we have

$$x(t) \approx x_P(t) = A\ \cos(\Omega t) + B\ \sin(\Omega t)$$

- Hence if Ω is "small", the mass will move very slowly, implying that $m\ddot{x}$ and $k\dot{x}$ will be much smaller than cx.

- In this "quasi-steady" regime, we expect the motion of the mass to be described (approximately!) by

$$c\,x(t) \approx F\ \cos(\Omega t).$$

“**Proof**”

- Check that
$$x(t) \approx \frac{F}{c}\cos(\Omega t)$$
is an approximate solution of
$$m\ddot{x} + k\dot{x} + cx = F\cos(\Omega t)$$
if Ω is small.

- The exact solution is
$$x(t) \approx x_P(t) = A\cos(\Omega t) + B\sin(\Omega t)$$
where
$$A = F\frac{c - m\Omega^2}{(k\Omega)^2 + (c - m\Omega^2)^2} \quad \to \frac{F}{c} \quad \text{as } \Omega \to 0,$$
and
$$B = F\frac{k\Omega}{(k\Omega)^2 + (c - m\Omega^2)^2} \quad \to 0 \quad \text{as } \Omega \to 0.$$

“Q.E.D.”

Exploiting small parameters: (Regular) perturbation expansions

An algebraic example:

$$x^2 + \varepsilon x - 1 = 0$$

$$x = -\frac{1}{2}\varepsilon \pm \sqrt{\left(\frac{\varepsilon}{2}\right)^2 + 1}$$

$$x(\varepsilon) = -\frac{1}{2}\varepsilon \pm \left(\frac{1}{4}\varepsilon^2 + 1\right)^{1/2}$$

Since ε is small consider its Taylor expansion about zero:

$$x(\varepsilon) = x(0) + \left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} \varepsilon + \frac{1}{2!} \left.\frac{d^2x}{d\varepsilon^2}\right|_{\varepsilon=0} \varepsilon^2 + \dots$$

$$x(0) = \pm 1$$

$$\frac{dx}{d\varepsilon} = -\frac{1}{2} \pm \frac{1}{4}\varepsilon\left(\frac{1}{4}\varepsilon^2 + 1\right)^{-1/2}$$

$$\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} = -\frac{1}{2}$$

$$\vdots$$

$$\left.\frac{d^2x}{d\varepsilon^2}\right|_{\varepsilon=0} = \pm\frac{1}{4}$$

So:

$$x(\varepsilon) = \pm 1 - \frac{1}{2}\varepsilon \pm \frac{1}{8}\varepsilon^2 \mp \frac{1}{128}\varepsilon^4 + \dots$$

This is a binomial series & it converges $|\varepsilon| < 2$.

Observation:

Have solution $\longrightarrow$ expand as power series in ε

Idea: Ansatz: Pose the solution as a power series.

Ansatz:

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

into equation

$$x^2 + \varepsilon x - 1 = 0$$

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)^2 +$$

$$+ \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) - 1 = 0$$

Expand & collect powers of ε

First term:

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)$$

$$= x_0^2 + \varepsilon 2 x_0 x_1 + \varepsilon^2 (2 x_0 x_2 + x_1^2)$$

$$+ \dots$$

$$\underline{(x_0^2 - 1)} + \varepsilon(2x_0x_1 + x_0) +$$
$$\varepsilon^2(2x_0x_2 + x_1^2 + x_1) + \dots = 0$$

Given that $|\varepsilon| \ll 1$
set the successive
coeffieients to zero:

$\underline{\varepsilon^0:}$ $x_0^2 - 1 = 0 \Rightarrow x_0 = \pm 1$

$\underline{\varepsilon^1:}$ $2x_0x_1 + x_0 = 0 \Rightarrow x_1 = -\frac{1}{2}$

$\underline{\varepsilon^2:}$ $2x_0x_2 + x_1^2 + x_1 = 0$

$$x_2 = \pm \frac{1}{8}$$

$$x(\varepsilon) = \pm 1 - \frac{1}{2}\varepsilon \pm \frac{1}{8}\varepsilon^2 + \dots$$

See above

Note the structure of the equations:

- Lowest-order equation (in powers of ε) is the original eqn. for $\varepsilon = 0$.
- Higher-order eqns. provide corrections via a systematic hierarchy of eqns.
- Eqn. itself is satisfied to increasing accuracy (in terms of powers of ε).