1

$$y y'' - 2(y')^2 + 2y' = 0$$

(auton.) <u>Note:</u> $y = C$ is soln

subst $u = y' \Longrightarrow y'' = u \frac{du}{dy}$

$$u \left(y \frac{du}{dy} - 2u + 2 \right) = 0$$

$u = 0$: $y = C$; the bracket $= 0$

⋮

$$\ln|u-1| = \ln(Dy^2) \qquad D > 0$$

<u>$u > 1$:</u>

$$u = 1 + Dy^2 = \frac{dy}{dx}, \qquad Dy^2 > 0$$

<u>$u < 1$</u>

$$\ln(1-u) = \ln Dy^2$$

$$u = 1 - Dy^2 = \frac{dy}{dx}$$

⋮

$$x + \hat{C} = \frac{1}{\sqrt{D}} \arctan(\sqrt{D}\, y)$$

$$y(x) = \frac{1}{\sqrt{D}} \tan(\sqrt{D}\, x + C)$$

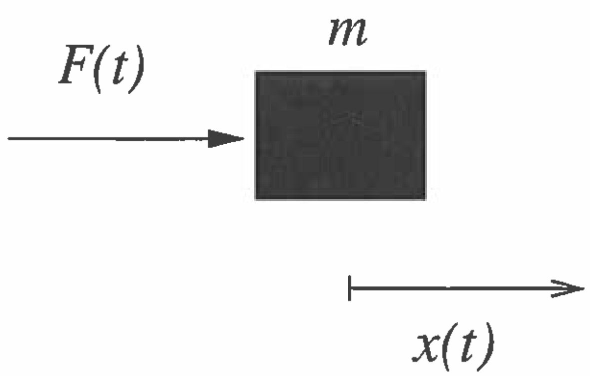
$\hookrightarrow \hat{C}\sqrt{D}$

Everything you always wanted to know about mechanical oscillators but were afraid to ask

- The first half of MATH10222 is not directly concerned with mechanics.
- However, mechanical systems provide nice illustrations of many of the phenomena that we have discussed (or will discuss) in a more abstract mathematical setting.

I. Newton's law for one-dimensional motion

- In words: "*The sum of all forces acting on a particle of mass m is equal to its mass times its acceleration*"

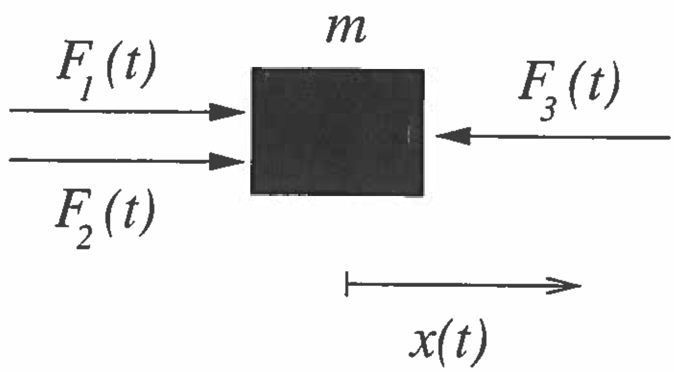


- Or, written as an equation:

$$m\frac{d^2x}{dt^2} = F(t)$$

I. Newton's law for one-dimensional motion (cont.)

- Here's an example with multiple forces



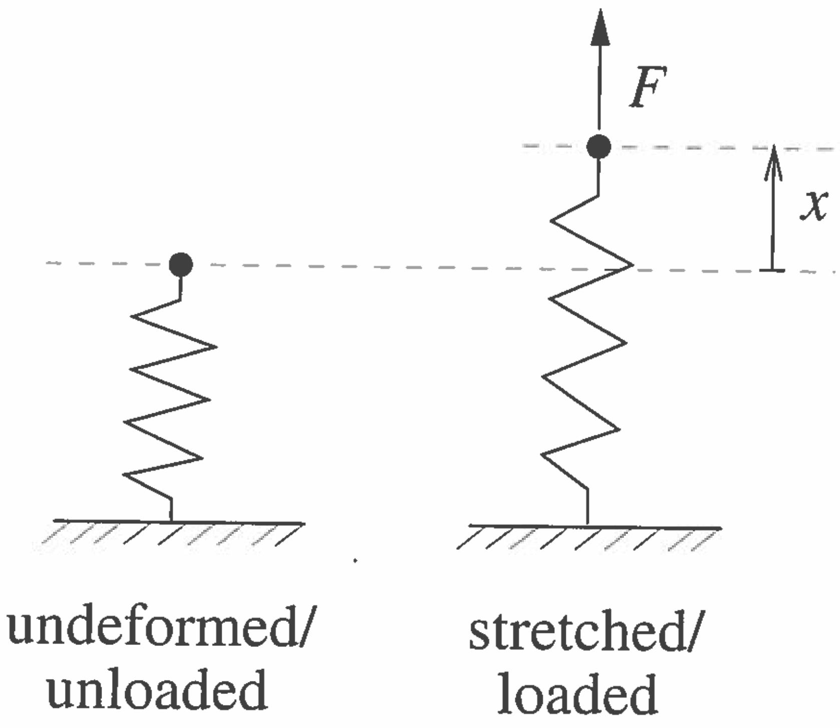
- In this case Newton's law becomes:

$$m\frac{d^2x}{dt^2} = F_1(t) + F_2(t) - F_3(t)$$

- Note the direction of the forces!

II. (Linearly) elastic springs

- Observation: When a spring is loaded by a force, F, its length increases by a certain amount, x, say.



- For a linearly elastic spring we have

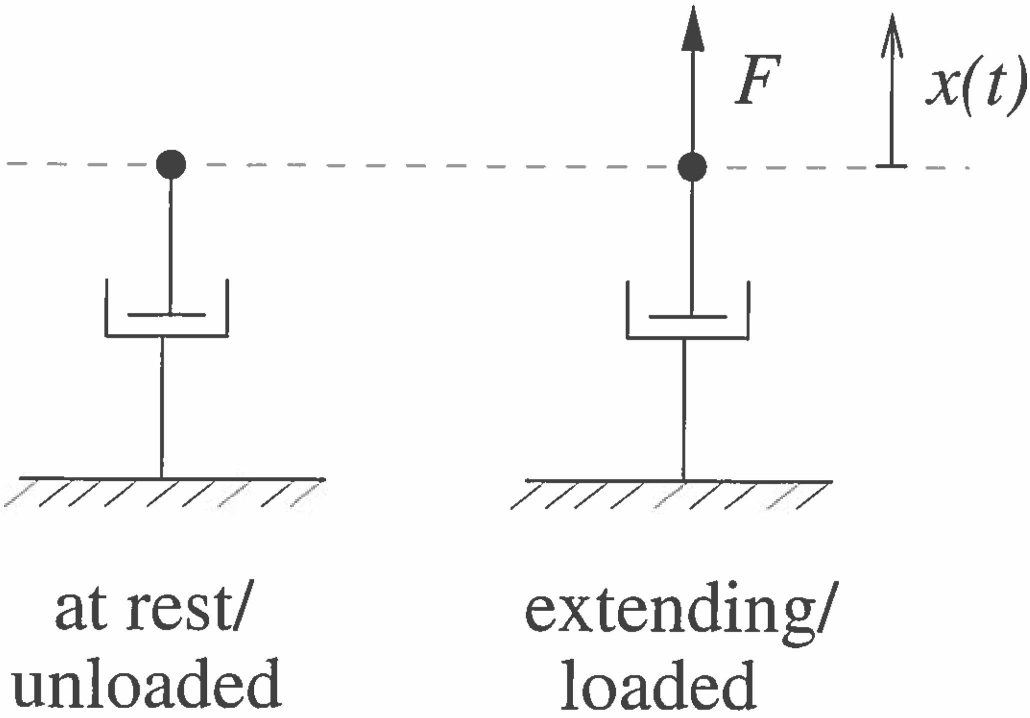
$$F = c\,x$$

where c is the "spring constant", a measure of its stiffness.

- Thus c indicates how strongly the spring resists its *static* extension.

III. (Linear) dampers

- Observation: When a damper is loaded by a force F its length increases at a rate dx/dt:



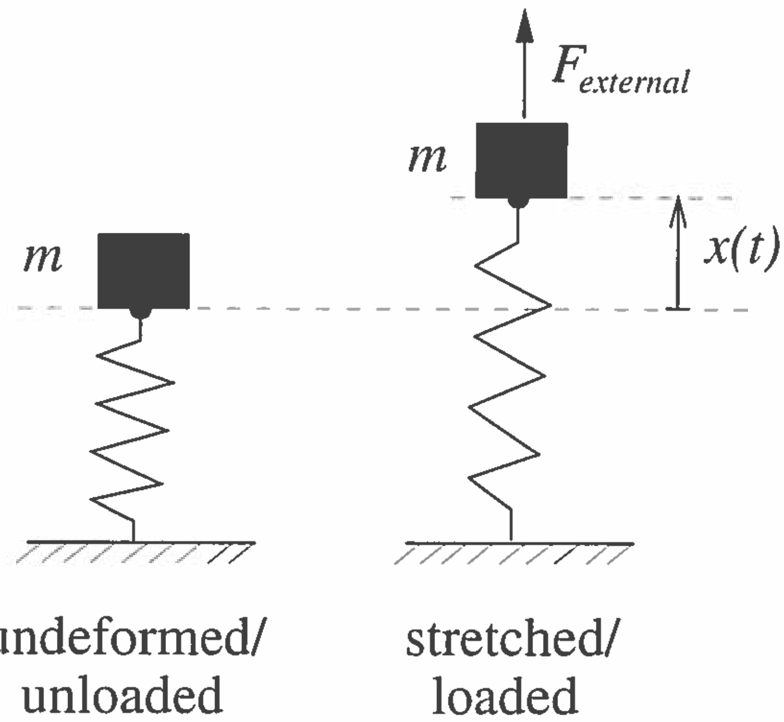
- For a linear damper we have

$$F = k\,\frac{dx}{dt}$$

where k is the "damping constant", a measure of how strongly the damper resists its *dynamic* extension.

IV. Putting it all together: "Action = Reaction"

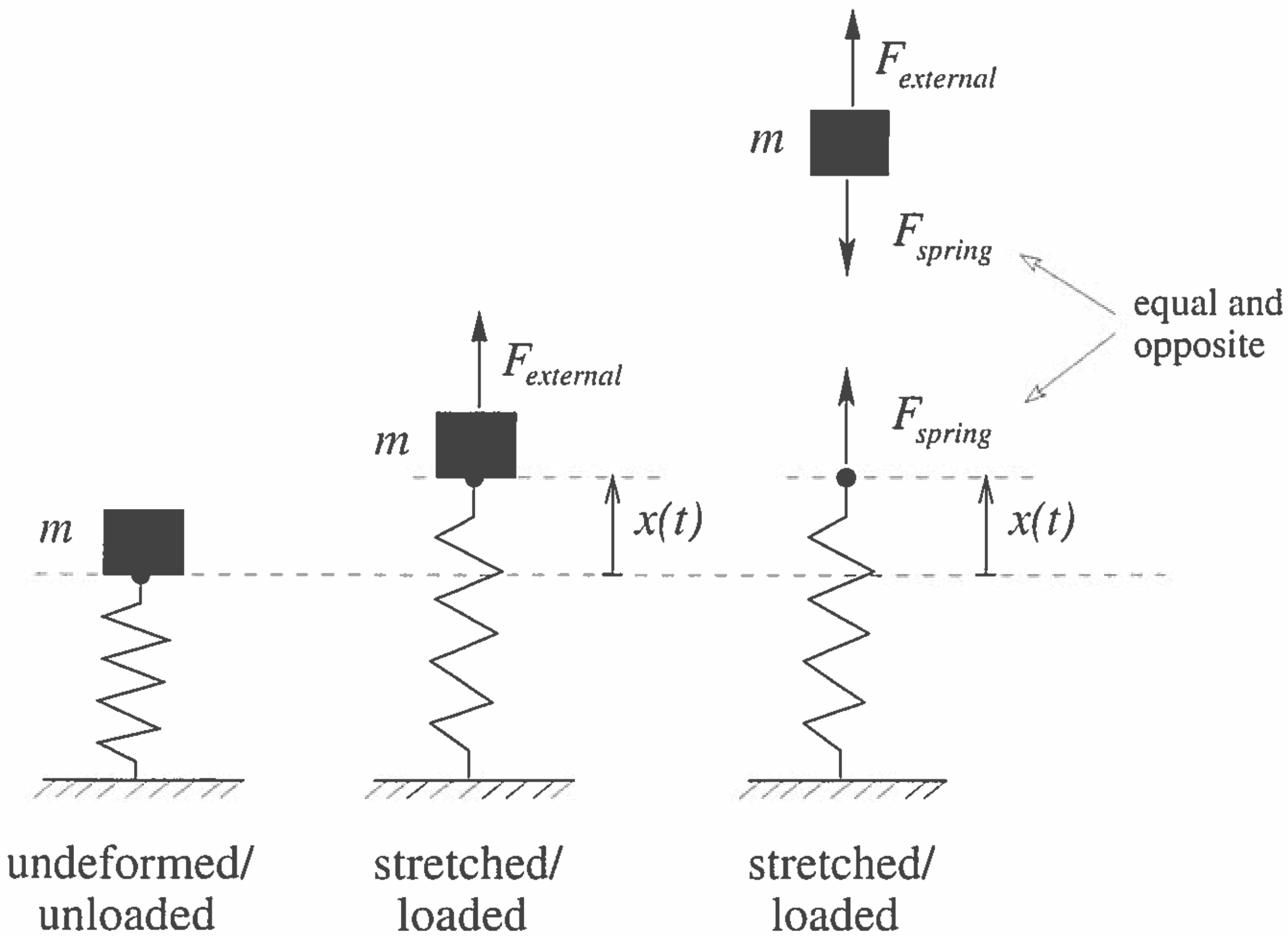
- Here is a mass m, attached to a spring of stiffness c, and loaded by a force, $F_{external}$.



- What is the equation of motion for the mass?
- Write down Newton's law for the mass.
- $\Longrightarrow$ What forces act on the mass?

IV. Putting it all together (cont.)

- "Action = Reaction": The spring pulls the mass and mass pulls the spring (in the opposite direction, obviously!):



- Thus Newton's law states

$$m\frac{d^2x}{dt^2} = F_{external} - F_{spring},$$

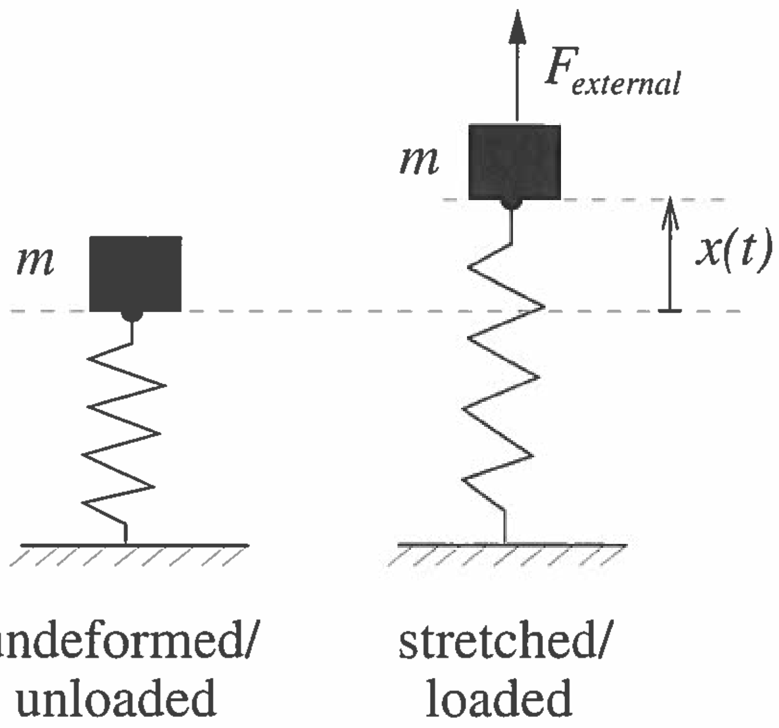
or, using what we've just learned about linear springs:

$$m\frac{d^2x}{dt^2} = F_{external} - cx.$$

IV. Putting it all together (cont.)

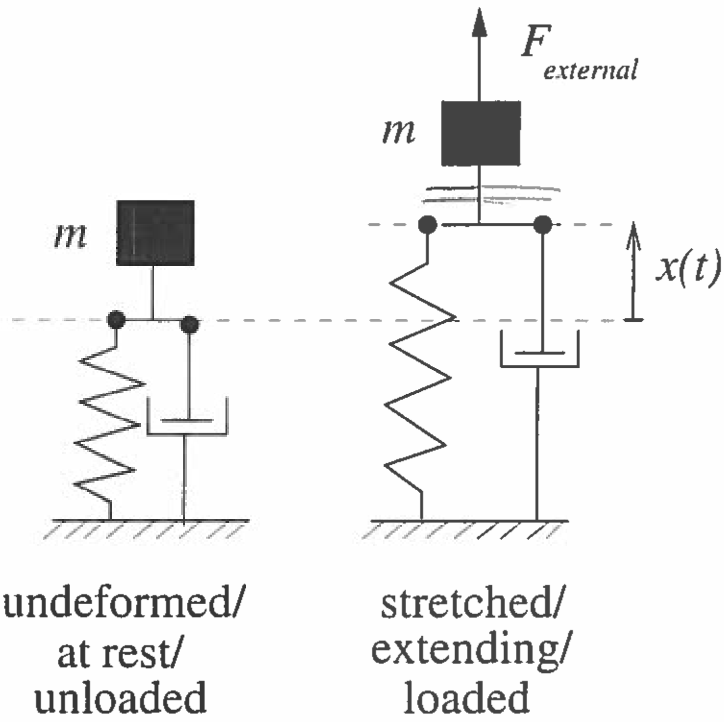
- Rewrite to the standard form of a second-order ODE for $x(t)$:

$$m\frac{d^2x}{dt^2} + cx = F_{external}.$$



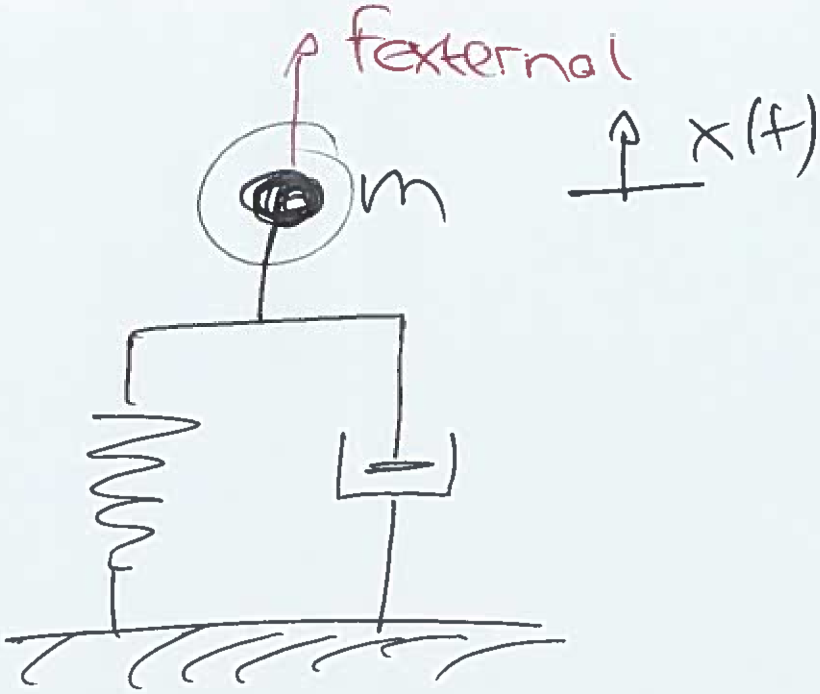
Exercise: Try it for yourself

- Here is a mass m, attached to a spring of stiffness c, and a damper (damping constant k), loaded by a force $F_{external}$.

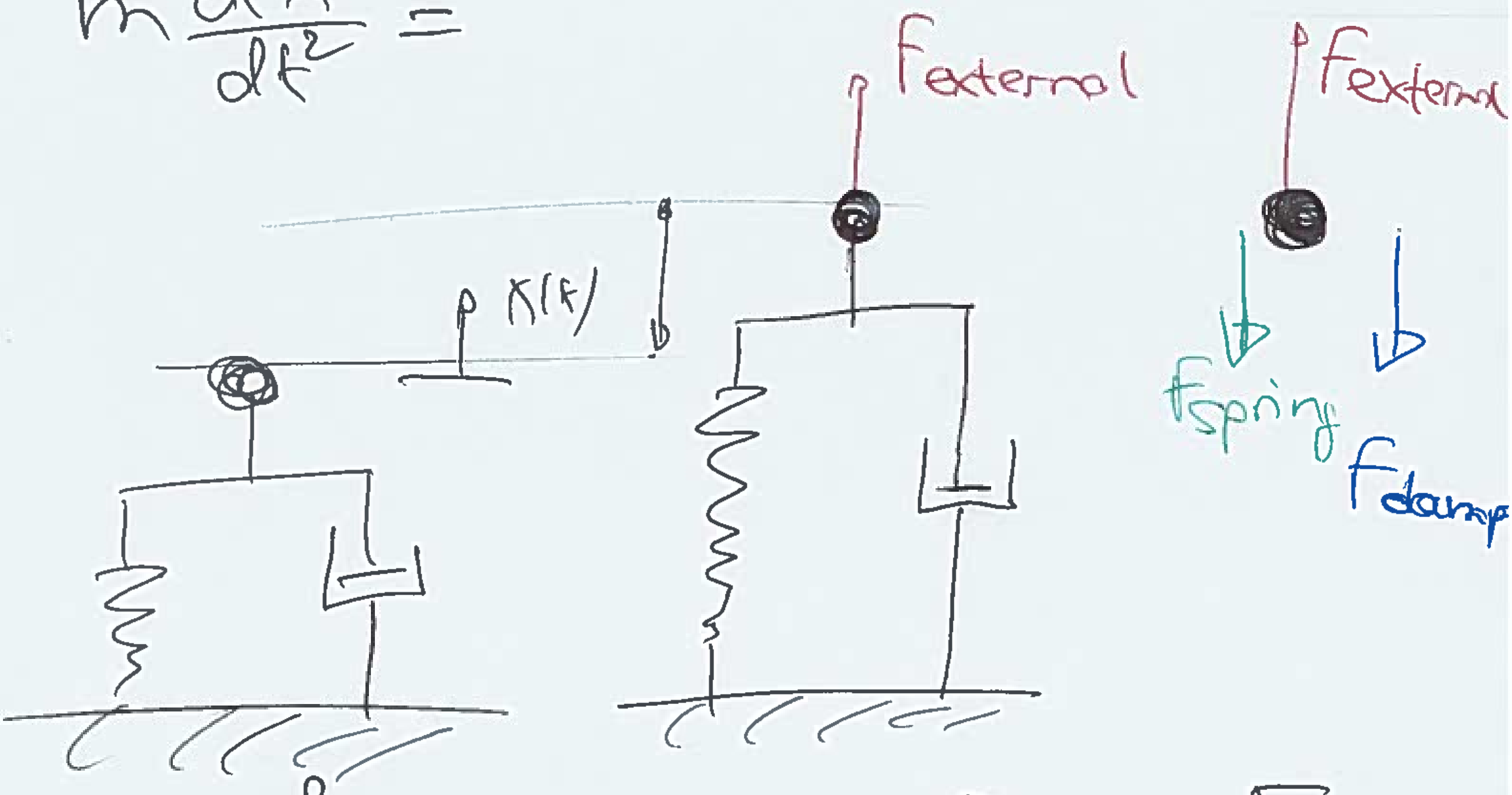


- Show that the equation of motion for the mass is

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + cx = F_{external}$$



$$m\frac{d^2x}{dt^2} =$$



$$m\frac{d^2x}{dt^2} = F_{external} - \underbrace{F_{spring}}_{c\,x(t)} - \underbrace{F_{damp}}_{k\frac{dx}{dt}}$$

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + c\,x = F_{external}(t)$$

Rewrite in standard form:

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = f(t)$$

$$\delta = \frac{k}{2m} > 0$$

$$\omega^2 = \frac{c}{m} > 0$$

$$f(t) = \frac{F_{external}(t)}{m}$$

IC: $x(t=0) = X_0$ initial position

$$\left.\frac{dx}{dt}\right|_{t=0} = V_0$$ initial velocity

The unforced case: $f(t) = 0$

(Eigenfrequencies)

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = 0$$

$$\lambda_{12} = -\delta \pm \sqrt{\delta^2 - \omega^2}$$

Four cases:

① Purely damped motion:

$\delta > \omega$

$$x(t) = A e^{(-\delta + \sqrt{\delta^2 - \omega^2})t} + B e^{(-\delta - \sqrt{\delta^2 - \omega^2})t}$$

both λ_{12} are negative:

$x \to 0$ as $t \to \infty$ without any oscillation

② Critically damped motion:

$\delta = \omega$

repeated roots: $\lambda_{12} = -\delta$

$$x(t) = A e^{-\delta t} + B t e^{-\delta t}$$

$x \to 0$ as $t \to \infty$ with at most one overshoot depending on ICs.

(3) Damped oscillations

$\delta < \omega$

$$\lambda_{12} = -\delta \pm i\sqrt{\omega^2 - \delta^2}$$

$$x(t) = e^{-\delta t}\left(A\cos\left(\sqrt{\omega^2-\delta^2}\,t\right) + B\sin\left(\sqrt{\omega^2-\delta^2}\,t\right)\right)$$

damped oscillation with frequency $\sqrt{\omega^2-\delta^2}$ whose amplitude decays $\sim e^{-\delta t}$.

$\frac{1}{\delta}$ is the timescale over which the oscillation decays.

(4) Undamped oscillation: $\delta = 0$

$$\lambda_{12} = \pm i\omega$$

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

Periodic undamped oscillation with frequency ω.

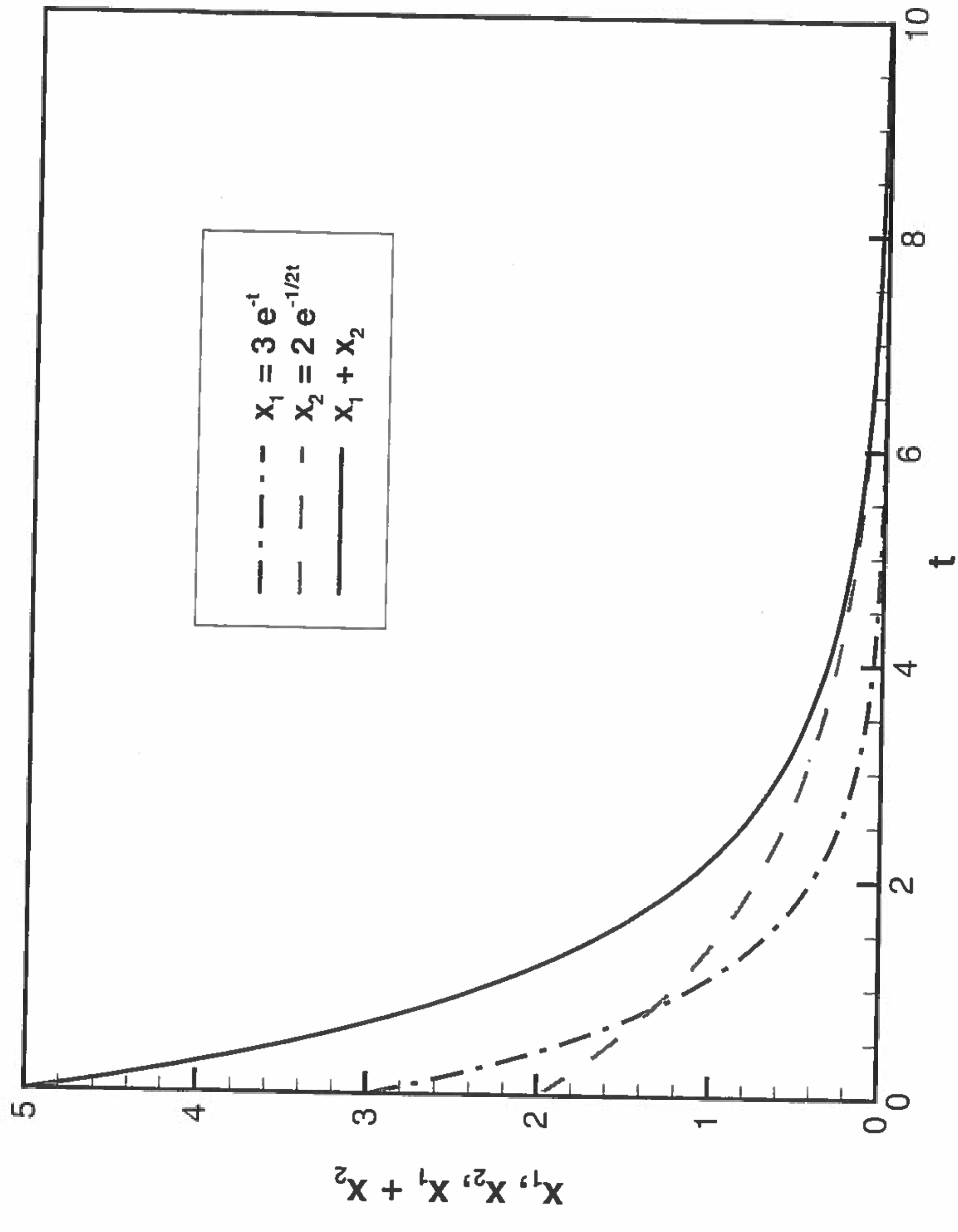


Figure 1: Illustration of a purely damped motion. The mass approaches its equilibrium position $x = 0$ monotonically.

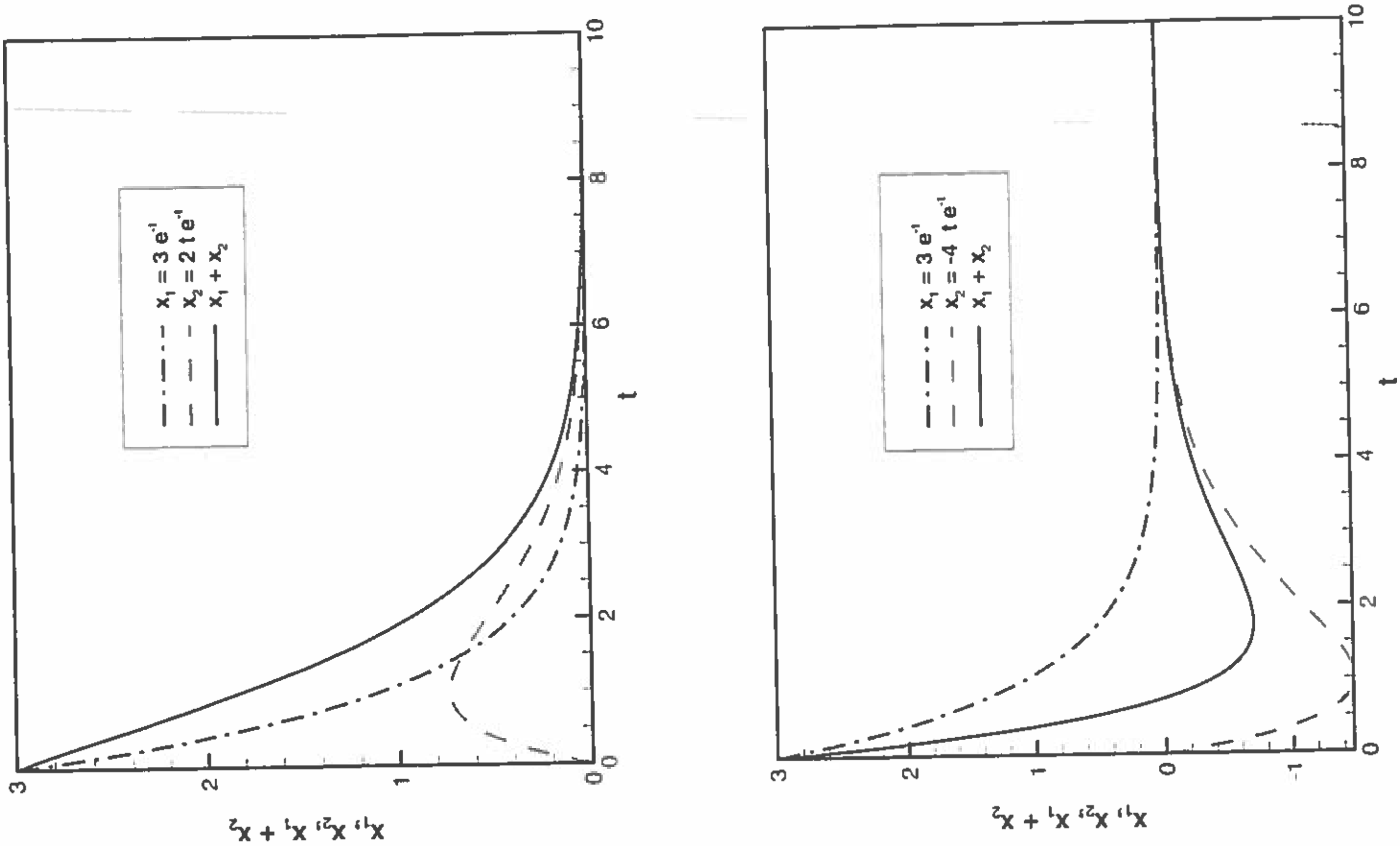


Figure 2: Illustration of critically damped motions. The mass approaches its equilibrium position, $x = 0$, with at most one "overshoot".

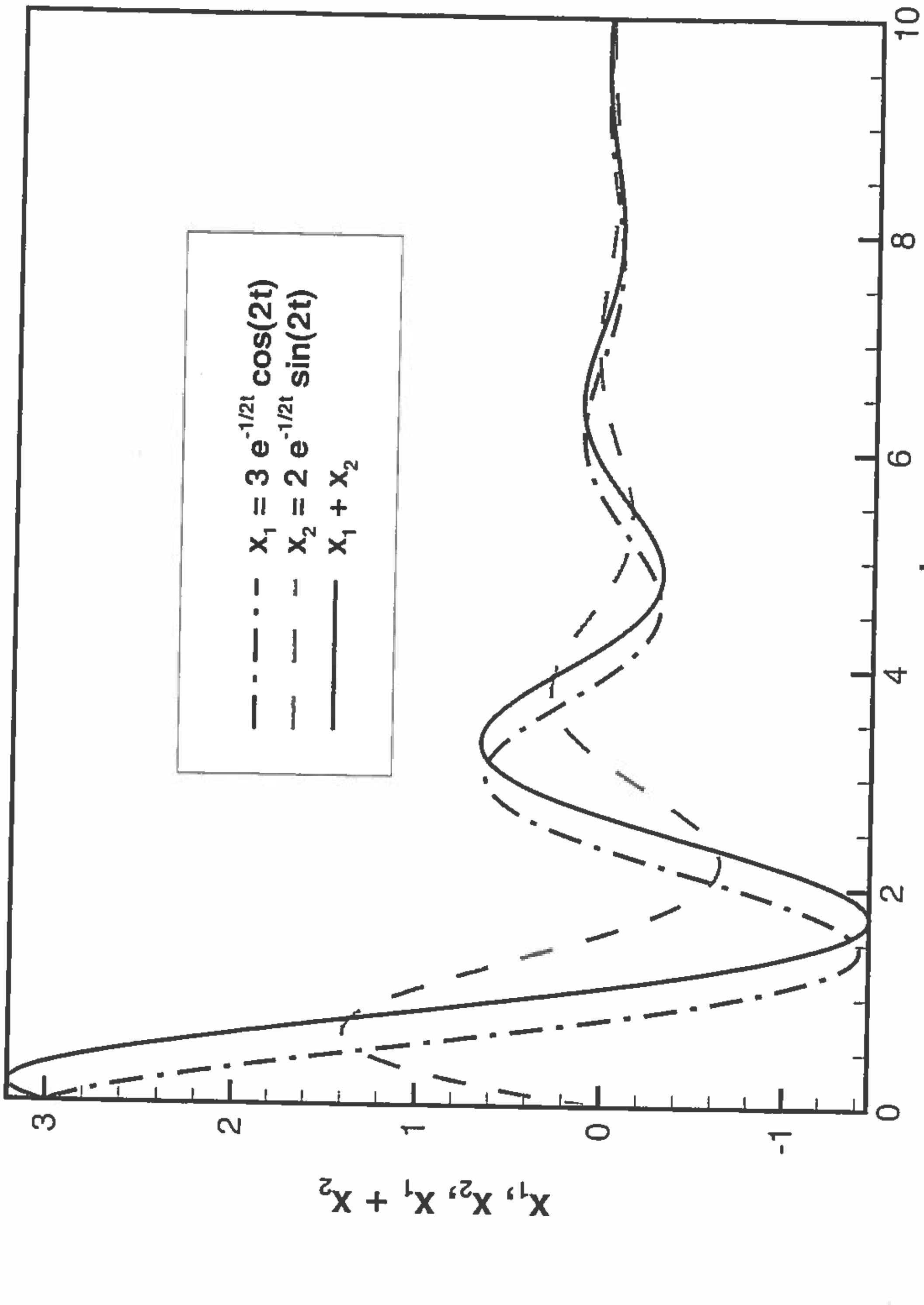


Figure 3: Illustration of a damped oscillation. The mass oscillates about its equilibrium position $x = 0$ and the amplitude of the oscillations decays exponentially.

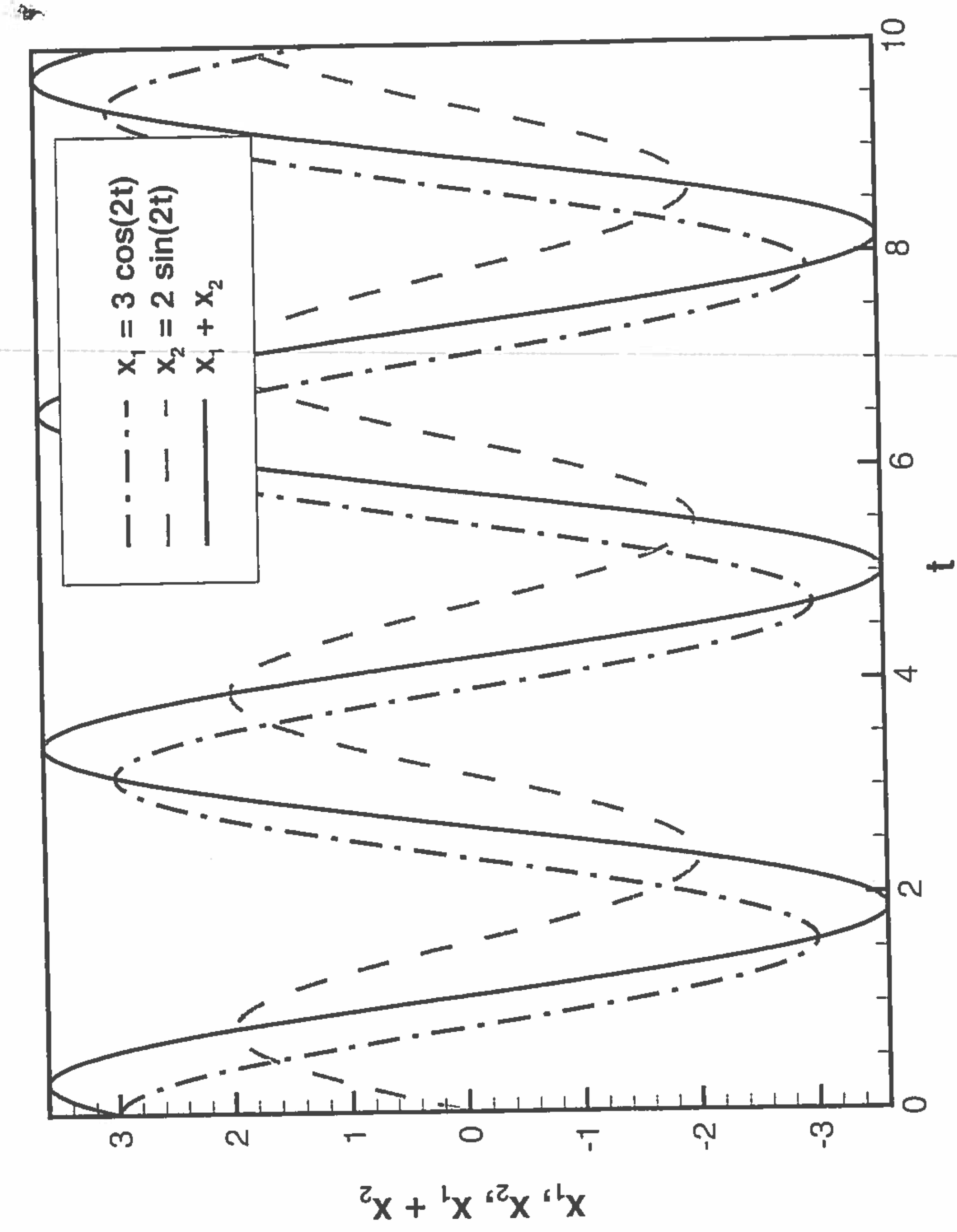


Figure 4: Illustration of an undamped oscillation. The mass performs harmonic oscillations about its equilibrium position $x = 0$.

Periodic forcing & resonance

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = f(t)$$

of particular interest: periodic forcing

$$f(t) = \hat{f}\sin(\Omega t) \quad \text{or} \quad \hat{f}\cos(\Omega t)$$

given

Can do both cases by considering

$$f(t) = \hat{f}e^{i\Omega t}$$

then extract real part of solution for $\cos(\Omega t)$ & imag. part for $\sin(\Omega t)$.

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = \hat{f}e^{i\Omega t}$$

Ansatz for particular soln:

$$x_p = X e^{i\Omega t}$$
$$\dot{x}_p = i\Omega X e^{i\Omega t}$$
$$\ddot{x}_p = -\Omega^2 X e^{i\Omega t}$$

into ODE

$$X e^{i\Omega t}\left(-\Omega^2 + 2\delta i\Omega + \omega^2\right) \stackrel{!}{=} \hat{f} e^{i\Omega t}$$

$$X = \frac{\hat{f}}{(\omega^2 - \Omega^2) + i\,(2\delta\Omega)}$$

is complex!

$$X = X_{real} + i\,X_{imag} = \underbrace{|X| e^{i\varphi}}_{\text{polar form}}$$

$$\varphi = \arg(X)$$

of most interest is

$$|X| = \frac{\hat{f}}{\sqrt{(\omega^2 - \Omega^2)^2 + (2\delta\Omega)^2}}$$

This is the amplitude of the oscillation in response to the forcing of magnitude $\hat{f}$