Existence and uniqueness theorem for 1st order ODEs

Consider the first-order ODE in its explicit form

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

subject to the initial condition

$$y(X) = Y, \tag{2}$$

where the constants X and Y are given.

Theorem

If $f(x, y)$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous functions of x and y in a region $0 < |x - X| < a$ and $0 < |y - Y| < b$, then there **exists** **exactly one** solution to the initial value problem defined by (1) and (2) in an interval $0 < |x - X| < h \leq a$.

Notes:

- The statement is easily generalised to higher-order ODEs.
- The theorem only provides a local statement!
- The statement only applies to initial value problems!
- The criteria listed are *sufficient* to ensure the existence of a unique solution but they are *not necessary*! $\Longrightarrow$ An IVP may still have a unique solution even if the conditions are violated.

A pretty weak statement then....

Existence and uniqueness theorem for *linear* 1st order ODEs

Consider the *linear* first-order ODE

$$\frac{dy}{dx} + p(x)\, y = q(x), \tag{3}$$

subject to the initial condition

$$y(X) = Y, \tag{4}$$

where the constants X and Y and the functions $p(x)$ and $q(x)$ are given.

Theorem

> If the functions $p(x)$ and $q(x)$ are continuous functions in an interval I, and if $X \in I$ then there **exists exactly one** solution to the initial value problem defined by (3) and (4) in the entire interval I.

Notes:

- The statement is again easily generalised to higher-order ODEs.
- The theorem provides a "much more global" statement. In fact, if the functions $p(x)$ and $q(x)$ are "well-behaved" (no jumps, singularities, etc.) the theorem guarantees the existence of a unique solution for $x \in \mathbb{R}$.
- However, the statement still only applies to initial value problems!

This is a much stronger statement and explains in part why (some) mathematicians love (only) linear problems.