MATH10222: EXAMPLE SHEET[1] IV

Questions for supervision classes

Hand in the solutions to questions 1a,c,d and 3. Attempt all other questions too and raise any problems with your supervisor. Question 1b should be straightforward. If you're a self-respecting mathematician you should find question 2 interesting.

1. **Inhomogeneous linear second-order ODEs with constant coefficients**

   (a) **Exploiting linearity**
   Find the general solutions of each of the following linear ODEs:

   i.
   $$\ddot{y} + 3\,\dot{y} + 2y = 4\,e^{2\,t}.$$

   ii.
   $$\ddot{y} + 3\,\dot{y} + 2\,y = 7.$$

   iii. Use the results just obtained to write down the general solution of the ODE
   $$\ddot{y} + 3\,\dot{y} + 2\,y = 4\,e^{2\,t} + 7.$$

   (b) **Using complex variables for trigonometric RHSs**
   Find the general solutions of

   i.
   $$\ddot{y} + 2\,\dot{y} + 2\,y = 10\,\cos t$$

   ii.
   $$\ddot{y} + 2\,\dot{y} + 2\,y = 10\,\sin t.$$

   [**Hint:** Use complex variables to do both in one go.]

   (c) **Degenerate and non-degenerate cases for RHSs of exponential form**
   Find the general solutions of the following ODEs. Watch out for degenerate cases!

   i.
   $$\ddot{y} + 3\,\dot{y} + 2\,y = 2\,e^{-t}$$

   ii.
   $$\ddot{y} + 4\,\dot{y} + 4\,y = e^{-2\,t}$$

   iii.
   $$\ddot{y} + 2\,\dot{y} + 2\,y = 5\,\cosh t$$

   iv.
   $$\ddot{y} + 3\,\dot{y} + 2\,y = 2\,\cosh t$$

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(d) **Degenerate and non-degenerate cases for polynomial RHSs**
Find the general solutions of the following ODEs. Watch out for degenerate cases!

i.
$$\ddot{y} + 3\,\dot{y} + 2\,y = 1 + t^2$$

ii.
$$\ddot{y} + 2\,\dot{y} = 1 + t^2$$

iii.
$$\ddot{y} = 1 + t^2$$

2. **Linear ODEs with non-constant coefficients: Euler's ODE**

(a) Homogeneous second-order Euler ODEs have the form

$$a\,t^2\,\ddot{y} + b\,t\,\dot{y} + c\,y = 0,$$

where a, b and c are constants. Explain why looking for solutions of the form $y \sim t^n$ is a promising ansatz.

(b) Use the ansatz to obtain the general solution of

$$t^2\,\ddot{y} + 2\,t\,\dot{y} - 2\,y = 0$$

and explain why your method works.

(c) Explain why the ansatz doesn't work for the ODE

$$t^2\,\ddot{y} - t\,\dot{y} + y = 0.$$

How would you obtain a second, linearly independent solution for this ODE?

3. **Non-linear ODEs with special properties**

(a) Solve the autonomous nonlinear ODE

$$y\,y'' = (y')^2.$$

(b) Solve the initial value problem

$$y'' = \frac{2x^2}{(y')^2}$$

subject to
$$y'(1) = 2^{1/3} \quad \text{and} \quad y(1) = 2^{-2/3}.$$

[**Hint:** Apply the initial conditions as soon as possible, otherwise the algebra gets messy...]