MATH10222: SOLUTIONS TO EXAMPLE SHEET[1] I

1. **Classifying ODEs**

   (a) **Nonlinear** because of the squared u in the second term on the LHS.
   Non-autonomous because the indepdendent variable, x, appears explicitly (on the RHS).
   First order because the highest derivative of the unknown function with respect to the independent variable is $u'(x)$ – a first derivative.

   (b) **Linear** because the ODE is linear in the unknown function, u, and its derivatives.
   Non-autonomous because the indepdendent variable, t, appears explicitly (in the term multiplying $u(t)$ on the LHS).
   Fourth order because the highest derivative of the unknown function with respect to the independent variable is d^4u/dt^4 – a fourth derivative.

   (c) **Nonlinear** because the ODE involves the sin of the unknown function, $\theta(t)$.
   Autonomous because the independent variable, t, does not appear explicitly.
   Second order because the highest derivative of the unknown function with respect to the independent variable is $\ddot{\theta}$ – a second derivative.

2. **Properties of ODEs**

   (a) False. $\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 2s\phi$ is a first order ODE. The highest derivative of the unknown function with respect to the independent variable is $\frac{\mathrm{d}\phi}{\mathrm{d}s}$ – a first derivative.

   (b) False. u'' and u' are evaluated at different values of the indepdendent variable, namely at x and at $x-1$, respectively, so $u''(x) + u'(x-1) = 1$ is not an ODE. (It's a delay differential equation – a different beast altogether).

   (c) False. $\frac{\mathrm{d}x}{\mathrm{d}y} + 5y^2x = 0$ is a linear ODE for $x(y)$.

   (d) False. One solution of $y'^2 + y^2 = 0$ is $y(x) = 0$.

   (e) False. $t^2\frac{\mathrm{d}^2t}{\mathrm{d}z^2} + 2t\frac{\mathrm{d}t}{\mathrm{d}z} + 2t = 0$ is an ODE for $t(z)$ and the indepdent variable, z does not occur explicitly.

3. **Solutions of ODEs; Boundary and Initial Value Problems**
   If
   $$y = A_1\,\mathrm{e}^x + A_2(1+x),$$
   then
   $$y' = A_1\,\mathrm{e}^x + A_2$$
   and
   $$y'' = A_1\,\mathrm{e}^x.$$
   Inserting this into the ODE $xy'' - (1+x)y' + y = 0$ gives
   $$xA_1\,\mathrm{e}^x - (1+x)(A_1\,\mathrm{e}^x + A_2) + A_1\,\mathrm{e}^x + A_2(1+x) = 0.$$
   The terms cancel and we obtain $0 = 0$, so $y = A_1\,\mathrm{e}^x + A_2(1+x)$ *is* a solution of the ODE.

[1] Any feedback to: *M.Heil@maths.man.ac.uk*

(a) $y(0) = A_1 + A_2 = 1$ and $y(1) = A_1\,e + 2A_2 = 1 \implies A_1 = \frac{-1}{e-2}$ and $A_2 = \frac{e-1}{e-2}$.

(b) $y(1) = A_1\,e + 2A_2 = 0$ and $y'(2) = A_1\,e^2 + A_2 = 0 \implies A_1 = 0$ and $A_2 = 0$.

(c) $y'(1) = A_1\,e + A_2 = e$ and $y'(1) = e = y(1) = A_1\,e + 2A_2 \implies A_1 = 1$ and $A_2 = 0$

Cases (a) and (b) are boundary value problems and (c) is an initial value problem.