

Basic ideas of perturbation methods: “Exploiting small parameters” and “Scaling”

Observation 1:

- ODEs (and hence their solutions!) typically contain some parameters, e.g.

$$m\ddot{x} + k\dot{x} + cx = F \cos(\Omega t)$$

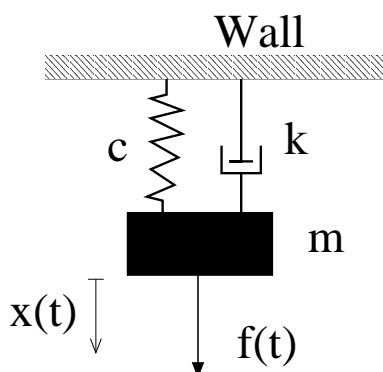
so

$$x = x(t) = x(t; m, k, c, \Omega).$$

- Often some of the problem’s parameters are “small”. How can we exploit this?
- Example:
 - Assume that we (only) know the solution of the above ODE for $k = 0$ (no damping).
 - What is the solution for “small” k ?

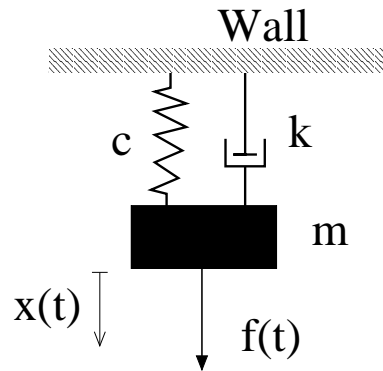
Observation 2:

- ODEs that model physical phenomena typically express balances (of forces, energies, currents, ...).
- Here's an example of a balance of forces:



$$\underbrace{m\ddot{x}}_{\text{inertial forces}} + \underbrace{k\dot{x}}_{\text{damping forces}} + \underbrace{cx}_{\text{spring forces}} = \underbrace{F \cos(\Omega t)}_{\text{applied external force}}$$

- In general, all terms in the ODE will make a significant contribution to the overall “balance”.
- However, there *may* be regimes in which the balance of terms is dominated by a balance between just a few (ideally two) terms, while the other terms only provide “negligible” contributions.
- The simplified equations (obtained by neglecting the small terms) are often much easier to solve than the full equations.
- We may [should!] then be interested in finding the effect that the “small” perturbations have on the solution.
- A seemingly trivial observation: You will need *at least* two terms to balance!

Example:

$$m\ddot{x} + k\dot{x} + cx = F \cos(\Omega t)$$

- We established earlier that

$$x(t) = x_P(t) + x_H(t)$$

where $x_H(t) \rightarrow 0$ very rapidly.

- Following the decay of the initial transients [described by $x_H(t)$] we have

$$x(t) \approx x_P(t) = A \cos(\Omega t) + B \sin(\Omega t)$$

- Hence if Ω is “small”, the mass will move very slowly, implying that $m\ddot{x}$ and $k\dot{x}$ will be much smaller than cx .
- In this “quasi-steady” regime, we expect the motion of the mass to be described (approximately!) by

$$cx(t) \approx F \cos(\Omega t).$$

“Proof”

- Check that

$$x(t) \approx \frac{F}{c} \cos(\Omega t)$$

is an approximate solution of

$$m\ddot{x} + k\dot{x} + cx = F \cos(\Omega t)$$

if Ω is small.

- The exact solution is

$$x(t) \approx x_P(t) = A \cos(\Omega t) + B \sin(\Omega t)$$

where

$$A = F \frac{c - m\Omega^2}{(k\Omega)^2 + (c - m\Omega^2)^2} \rightarrow \frac{F}{c} \quad \text{as } \Omega \rightarrow 0,$$

and

$$B = F \frac{k\Omega}{(k\Omega)^2 + (c - m\Omega^2)^2} \rightarrow 0 \quad \text{as } \Omega \rightarrow 0.$$

“Q.E.D.”

Observation 3a:

- Coefficients occurring in ODEs that model physical phenomena have dimensions!
- The dimensions of all terms must be (are!) consistent.

$$m\ddot{x} + k\dot{x} + cx = F \cos(\Omega t)$$

$$\underbrace{m} \underbrace{\ddot{x}} + \underbrace{k} \underbrace{\dot{x}} + \underbrace{c} \underbrace{x} + = \underbrace{F} \cos(\underbrace{\Omega} \underbrace{t})$$

$$\underbrace{m}_{\text{kg}} \underbrace{\ddot{x}}_{\text{m/sec}^2} + \underbrace{k}_{?} \underbrace{\dot{x}}_{\text{m/sec}} + \underbrace{c}_{\text{N/m}} \underbrace{x}_{\text{m}} + = \underbrace{F}_{\text{N}} \cos\left(\underbrace{\Omega}_{1/\text{sec}} \underbrace{t}_{\text{sec}}\right)$$

- What's the dimension of k ? For dimensional consistency:

$$[k] = \text{N}/(\text{m}/\text{sec})$$

or (since $N = \text{kg m}/\text{sec}^2$; see $m\ddot{x}$)

$$[k] = \text{k}/\text{sec}$$

- The arguments of all functions (e.g. $\cos \Omega t$) are dimensionless!

Observation 3b:

- The solution tends to depend on ratios of dimensional coefficients.
- The ratios provide an indication of:
 1. The relative size of the physical effects* represented by the corresponding terms.

$$m\ddot{x} + k\dot{x} + cx = f \cos(\Omega t)$$

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = F \cos(\Omega t)$$

where

$$\delta = \frac{k}{2m} = \frac{\text{“Damping forces”}}{\text{“Inertia”}}$$

and

$$\omega^2 = \frac{c}{m} = \frac{\text{“Spring forces”}}{\text{“Inertia”}}.$$

2. Time/length-scales over which the relevant phenomena occur. E.g.

$$x(t) = e^{-\delta t} \left(A \cos(t\sqrt{\omega^2 - \delta^2}) + B \sin(t\sqrt{\omega^2 - \delta^2}) \right),$$

showing that

$\implies 1/\delta$ is a representative timescale over which the oscillations decay.

$\implies 1/\omega$ is a representative timescale for the undamped oscillation.

*: **Disclaimer:** Statement 1 is a bit too simple-minded – we might (!) have time to come back it...

Observations about Observations 1, 2 and 3

- The approach outlined above exploits *additional* knowledge about the problem.
- You will either have such knowledge *a priori* or you can make certain (hopefully plausible) assumptions about certain properties of the solution.
- In the latter case, you'll have to check the consistency of your assumptions when you're done. For instance:
 - Assume the the solution is such that certain terms in the ODE are small.
 - Neglect the small terms in the ODE and solve.
 - Check afterwards that the terms that were *assumed* to be small are *actually* small.
- The approach tends to produce approximate solutions of the ODE that are valid only in certain “*regions of parameter space*”, e.g. for small forcing frequencies Ω , small damping k , etc.
- This is often more useful than having an exact (but horrendously complicated) closed-form solution that is valid for all parameter values.