“Multinomial expansions”

- One tedious task that one tends to face regularly when using perturbation methods is that of raising a power series in ϵ to some integer power

$$S = \left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + ...\right)^n, \tag{1}$$

and collecting the terms multiplied by the same power of ϵ, i.e. re-writing S in the form

$$S = S_0(x_0) + \epsilon\, S_1(x_0, x_1) + \epsilon^2\, S_2(x_0, x_1, x_2) + ... \tag{2}$$

where the functions $S_i(x_0, x_1, ...)$ do not depend on ϵ.

- Formally, the expansion of S may be obtained by using the “multinomial series” (a generalisation of the binomial series) as

$$\left(a_1 + a_2 + ... + a_k\right)^n = \sum_{\substack{n_1, n_2, n_3, ..., n_k \in \mathbb{N}_0 \\ n_1 + n_2 + ... + n_k = n}} \frac{n!}{n_1!\, n_2! \,...\, n_k!}\, a_1^{n_1}\, a_2^{n_2} ... \, a_k^{n_k}$$

see, e.g. `http://mathworld.wolfram.com/MultinomialSeries.html`

- However, we usually only need the first few terms in (2) for low-ish powers of n. Here they are:

$$\begin{aligned}
\left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + ...\right)^2 &= \left(x_0^2\right) + \epsilon\left(2x_0\, x_1\right) + \epsilon^2\left(x_1^2 + 2x_0\, x_2\right) + ... \\
\left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + ...\right)^3 &= \left(x_0^3\right) + \epsilon\left(3x_0^2\, x_1\right) + \epsilon^2\left(3x_0 x_1^2 + 3x_0^2\, x_2\right) + . \\
\left(x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + ...\right)^4 &= \left(x_0^4\right) + \epsilon\left(4x_0^3\, x_1\right) + \epsilon^2\left(4x_0^3 x_2 + 6x_0^2\, x_1^2\right) + .
\end{aligned}$$

- **Exercise:** Convince yourself that you understand how these terms arise. **Hint:** Either use the multinomial series given above, or write S explicitly as a product of n power series [e.g. for $n = 2:\ \ S = (x_0 + \epsilon\, x_1 + ...)(x_0 + \epsilon\, x_1 + ...)$] and inspect which combination of terms gives rise to what powers of ϵ.

- **Relax!** In an exam these expressions would be provided!