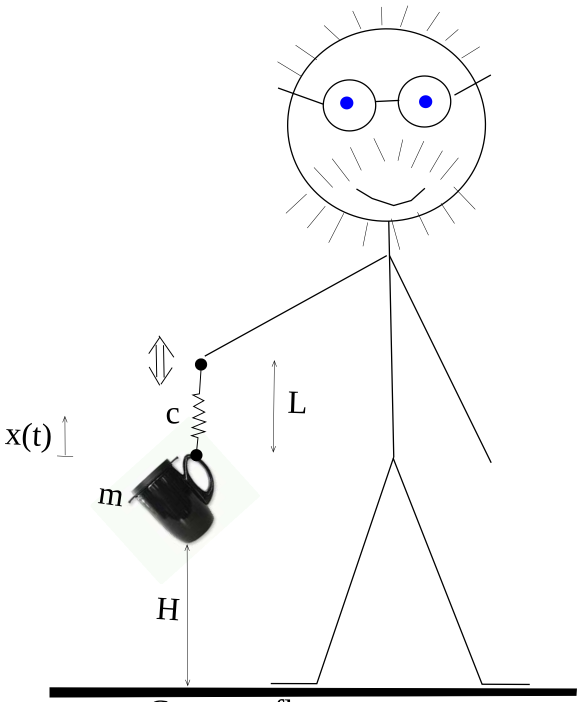
MATH10222: EXAMPLE SHEET[1] V

Questions for supervision classes

Hand in the solutions to questions 1a,b,c(i,ii) and 3. Discuss any problems with your supervisor. The remaing questions are more advanced (and therefore more interesting!). If you like mechanics, have a go at them too – make sure you follow the hints!

1. **Applications of second-order ODEs: "Resonance" or "How to destroy a coffee mug..."**



To demonstrate the phenomenon of resonance your lecturer takes his coffee mug (mass m) to the lecture theatre and suspends it from a piece of rubber string (spring stiffness c). Fig. 1 shows the initial configuration: The mug is at rest, the point at which the rubber string is attached to its handle is located at $x = 0$, and the upper end of the rubber string is located at $x = L$. The lowest point of the (fragile and precious) mug is located at a distance H from the concrete floor.

Displaying immense technical skill, your lecturer now subjects the upper end of the rubber string to periodic vertical displacements so that its position is given by

$$\widehat{x}(t) = \begin{cases} L & \text{for } t < 0 \\ L + A\,\sin(\Omega t) & \text{for } t \geq 0. \end{cases}$$

[1]Any feedback to: *M.Heil@maths.man.ac.uk*

(a) Consider the balance of forces acting on the mug to show that for $t \geq 0$ its motion is governed by the ODE

$$m\ddot{x} + cx = cA\sin(\Omega t),$$

where $x(t)$ is the vertical displacement of the mug's handle from its initial position $x = 0$. [Hint: Work out the length of the rubber string at an instant when the handle has moved to $x = x(t)$ while the upper end of the rubber string has been displaced to $x = \widehat{x}(t)$. Recall that the force generated by an elastic spring is given by the change of its length times its stiffness c.]

(b) State the initial conditions to be applied at $t = 0$ and solve the initial value problem for the case when $\Omega < \omega = \sqrt{c/m}$. Show that Dr Heil's mug is safe as long as $A < H\left(1 - \left(\frac{\Omega}{\omega}\right)\right)$.

(c) Dr Heil is away at a conference and one of his colleagues gives the MATH10222 lecture for him. The helpful colleague happens to be a pure mathematician who doesn't like mechanics and therefore fails to appreciate the practical importance of the resonance phenomenon. He performs the experiment exactly as instructed (and as described above), but he rather foolishly decides to oscillate the mug with the frequency $\Omega = \omega = \sqrt{c/m}$. We wish to determine how long it takes before Dr. Heil's precious mug is shattered on the floor.

   i. Re-solve the initial value problem for the "resonant" case and state the equation that determines the time t_{shatter} at which the mug hits the floor.

   ii. The condition derived in 1(c)i is a transcendental equation that cannot be solved in closed form. Use a plot of the height of the mug above the concrete floor, $x(t) + H$, to determine an approximate value for t_{shatter} for the special case $\Omega = \omega = \pi\,\text{sec}^{-1}$, $A = 0.01\,\text{m}$ and $H = 1.5\,\text{m}$.

   iii. Exploit the fact that $A \ll H$ to obtain an approximate value for t_{shatter}. [Hint: Compare the sizes of the various terms in the "shatter condition" close to the moment of impact.]

2. **Applications of second-order ODEs: The "Dead Cat Bounce"**
The "Dead Cat Bounce" is a well-known phenomenon on the stock market and is often observed when the share price of a badly-performing company tumbles. Just before the stock price hits zero, the price often shows some temporary recovery and increases again. The financial analyst (who looks after *your* investment) now has to address a crucial question:

   - Is this a sign of long-term recovery and should he (or she!) therefore buy lots of shares, at rock-bottom prices, and thus make an enormous profit as the stock price continues to rise.
   - Is the rise in the share price the infamous "Dead Cat Bounce" and therefore irrelevant – the fact that a cat bounces when it hits the pavement after being flung out of a window on the 50th floor of a Canary Wharf tower is not an indication that it's going to do well in the future...

   To avoid the needless suffering of furry animals, we shall develop a simple mathematical model of dead-cat-bouncing to analyse the factors that really affect the cat's behaviour following its impact.

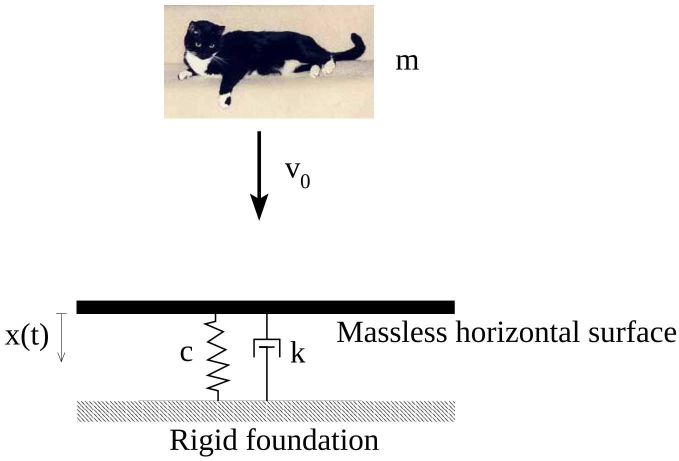


Figure 1: Model of a bouncing cat.

Fig. 1 shows the model: The cat (of mass $m > 0$) is subject to gravity which exerts a downward force of magnitude mg ($g > 0$ is the gravitational acceleration, a known constant). When the cat hits the floor (at time $t = 0$), its downward velocity is v_0. We model the floor as a massless horizontal surface, mounted on an elastic support (spring stiffness $c > 0$) and we assume that there is some damping (damping constant $k > 0$). Just before the impact, the floor is located at $x = 0$, it is at rest, and the elastic spring is undeformed.

(a) Use Newton's law ("the sum of all forces acting on the cat is equal to the product of its mass and acceleration") to formulate the initial value problem that describes the cat's position as a function of time, $x(t)$, for $t \geq 0$. Thus show that (at least the early stages of) the cat's motion are described by the ODE

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + cx = m\,g. \tag{1}$$

(b) State under what conditions the ODE provides a valid description of the cat's motion. [Hint: The floor exerts a force

$$F_{floor} = k\frac{dx}{dt} + cx,$$

onto the cat, where $F_{floor} > 0$ if the force acts upwards. Assuming that the cat is not glued to the floor (It's not!), can the floor exert any downward forces onto the cat?]. Explain how this condition provides a criterion that allows us to determine if and when the cat "bounces" – a bounce being defined as phase during which the cat lifts off the floor.

(c) Show that the condition is satisfied in the period immediately after the cat's impact. [Hint: Use the initial conditions to establish that $F_{floor}(t = 0) \geq 0$, showing that the ODE is valid at the moment of impact. Then use a Taylor expansion of $F_{floor}(t)$ for small positive values of t to show that $F_{floor}(t) \geq 0$ for at least a short while ($0 < t < \Delta t$, say) after the impact. Note: Having established that the ODE is valid at $t = 0$, you can use it to obtain an expression for $d^2x/dt^2|_{t=0}$.]

(d) Assuming that the ODE (1) *is* valid, and that $c/m = \omega^2 > [k/(2m)]^2 = \delta^2$, solve the initial value problem and determine the cat's trajectory. What is the cat's ultimate equilibrium position, $\lim_{t\to\infty} x(t)$?

(e) Use MATLAB (or your favourite plotting package) to investigate how variations in the damping constant δ affect the cat's motion. [Plot the solution $x(t)$ and the force $F_{floor}(t)$ for $v_0 = 10\,\text{m/sec}$, $g = 9.81\,\text{m/sec}^2$, $\omega = \pi\,\text{sec}^{-1}$ and $\delta = 0, 1, 2, 3\ \text{sec}^{-1}$, say]. Does an increase in δ encourage or discourage cat-bouncing?

(f) Show that the cat *will* bounce if there is no damping ($k = \delta = 0$).

3. **Motivating the use of scaling arguments to simplify ODEs**

We know that solutions of the ODE

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + cx = F\,\cos(\Omega t) \tag{2}$$

have the form

$$x(t) = x_P(t) + x_H(t)$$

where the "transient solution" $x_H(t)$ decays very rapidly. Furthermore, a particular solution $x_P(t)$ is known to have the form

$$x_P(t) = A\,\cos(\Omega t) + B\,\sin(\Omega t).$$

Guided by this form of the solution, we argued in the lecture that for small values of Ω it should be possible to obtain a good approximation for $x_P(t)$ from the simplified equation

$$c\,x(t) = F\,\cos(\Omega t), \tag{3}$$

the argument being that for sufficiently small Ω, the terms $m\,d^2x/dt^2$ and $k\,dx/dt$ in (2) can be neglected against the term $c\,x(t)$. We checked the "validity" of this approach by comparing the solution of (3) against the limiting form of the exact solution for x_P, obtained from (2), in the limit $\Omega \to 0$.

Repeat this analysis for the case of large Ω. Which term in (2) do you expect to become dominant as $\Omega \to \infty$? State the simplified equation and show that its solution

$$x_P(t) = -\frac{F}{m\Omega^2}\,\cos(\Omega t)$$

agrees with the limit of the exact solution of (2) as $\Omega \to \infty$.