MATH10222: EXAMPLE SHEET[1] III

Questions for supervision classes

Hand in the solutions to questions 1, 2a,b and 3a-f. [Feel free to skip the application of the initial conditions in question 3 once you're sure you know how it works. By the end of question 3 you should (i) be pretty bored with homogeneous constant-coefficient ODEs, and (ii) be able to do them in your sleep!] Attempt all other questions too. Questions 2c and 4 should be pretty straightforward; question 5 provides a more detailed (and interesting!) analysis of the "repeated roots" case.
As always, raise any problems with your supervisor.

1. **Existence and uniqueness for linear second-order ODEs**

   (a) Does the initial value problem

   $$x^2\, y'' - 2\, x\, y' + 2\, y = 0,$$

   subject to

   $$y(x = 1) = 1 \quad \text{and} \quad y'(x = 1) = 2$$

   have a unique solution? If so, specify the interval in which the solution is guaranteed to exist.

   (b) Does the initial value problem

   $$\ddot{x} - 2\,\frac{1}{t}\,\dot{x} + 2\,\frac{1}{t^2}\,x = 0,$$

   subject to

   $$x(t = -1) = 1 \quad \text{and} \quad \dot{x}(t = -1) = 2$$

   have a unique solution? If so, specify the interval in which the solution is guaranteed to exist.

   (c) Does the initial value problem

   $$\ddot{y} + \Omega^2\, y = 0,$$

   subject to

   $$y(t = 0) = 0 \quad \text{and} \quad \dot{y}(t = 0) = 0,$$

   where Ω is a given constant, have a unique solution? If so, specify the interval in which the solution is guaranteed to exist. Can you spot the solution without doing any calculations?

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[1] Any feedback to: *M.Heil@maths.man.ac.uk*

(d) Now consider the same ODE as in the previous example, but in the context of the *boundary* value problem

$$\ddot{y} + \Omega^2\, y = 0,$$

subject to

$$y(t = 0) = 0 \quad \text{and} \quad y(t = 1) = 0$$

where Ω is a given constant. Recall that the existence and uniqueness theorem does not apply to boundary value problems. Show that the solution of the initial value problem of question 1c is also a solution of the above boundary value problem, demonstrating the existence of *a* solution. Is it possible that there are other solutions? [Hint: Consider the special cases $\Omega = \pi, 2\pi, ...$]

2. **Linear and nonlinear second-order ODEs**

(a) Verify that $y_1(t) = t$ and $y_2(t) = t^2$ are linearly independent solutions of the ODE

$$t^2\, \ddot{y} - 2\, t\, \dot{y} + 2\, y = 0,$$

then write down its general solution.

(b) Verify that $y_1(t) = e^t$ and $y_2(t) = e^{2\,t}$ are linearly independent solutions of the ODE

$$y\, \ddot{y} - (\dot{y})^2 = 0,$$

but that $y = A\, y_1(t) + B\, y_2(t)$, where A and B are arbitrary constants, is not a solution. Explain why.

(c) Prove the statement that if $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous linear ODE

$$y'' + p(x)\, y' + q(x)\, y = 0,$$

then the linear combination $A\, y_1(x) + B\, y_2(x)$ is also a solution, for *any* values of the constants A and B.

3. **Homogeneous linear ODEs with constant coefficients**

Solve the following initial value problems:

(a) $\ddot{y} - 5\,\dot{y} + 4\, y = 0$ subject to $y(0) = 0, \quad \dot{y}(0) = 1.$

(b) $\ddot{y} + 4\, y = 0$ subject to $y(0) = 1, \quad \dot{y}(0) = 0.$

(c) $\ddot{y} - y = 0$ subject to $y(0) = 1, \quad \dot{y}(0) = 0.$

(d) $\ddot{y} + 4\,\dot{y} + 4\, y = 0$ subject to $y(0) = 1, \quad \dot{y}(0) = -2.$

(e) $\ddot{y} - 2\,\dot{y} + 3\, y = 0$ subject to $y(0) = 0, \quad \dot{y}(0) = \sqrt{2}.$

(f) $\ddot{y} = 0$ subject to $y(0) = 1, \quad \dot{y}(0) = -2.$

4. **The real form of the fundamental solutions in the case of complex conjugate roots of the characteristic polynomial**

Consider the fundamental solutions of the homogeneous ODE

$$y'' + p\, y' + q\, y = 0,$$

where the constants p and q are such that $q > (p/2)^2$. In the lecture we showed that the general solution of the ODE was given by

$$y(x) = e^{\mu x}\left(\widehat{A}\, e^{i\omega x} + \widehat{B}\, e^{-i\omega x}\right), \tag{1}$$

where $\omega = \sqrt{q - (p/2)^2}$ and $\mu = -p/2$. If we are only interested in real solutions, the constants $\widehat{A}$ and $\widehat{B}$ obviously have to be complex. In the lecture we had argued (rather convincingly, but indirectly) that it must possible to re-write the real solution in the form

$$y(x) = e^{\mu x}\left(A\, \cos(\omega x) + B\, \sin(\omega x)\right).$$

where $A, B \in \mathbb{R}$.

Prove this by "brute force" calculation. [**Hint:** Write $\widehat{A}$ and $\widehat{B}$ in terms of their real and imaginary parts, $\widehat{A} = \alpha + i\beta$ and $\widehat{B} = \gamma + i\delta$, say, where the constants $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Insert into (1) and expand, then set the imaginary part of the resulting expression to zero.]

5. **The form of the solution for repeated roots – "reduction of order"**
The characteristic polynomial for the homogeneous linear ODE

$$\ddot{y} + 2\, k\, \dot{y} + k^2\, y = 0, \tag{2}$$

has a repeated root $\lambda = -k$. One of the two fundamental solutions is therefore given by $y_1(t) = e^{-kt}$. We demonstrated in the lecture (by "brute force" checking) that $y_2(t) = t\, e^{-kt}$ is a second, linearly independent solution. What motivated your lecturer to suggest this as a possible solution?[2]

To solve this mystery, we will now demonstrate a systematic way of constructing a second solution, $y_2(t)$, to a homogeneous, second-order linear ODE if one solution, $y_1(t)$, is already known. The method (known as the "reduction of order") is to look for a solution of the form $y_2(t) = g(t)\, y_1(t)$, where $g(t)$ is an unknown function. Inserting this ansatz into the second-order ODE produces a first-order linear ODE for $\dot{g}(t)$ that can be integrated with standard methods (e.g. using the integrating factor method). Have a look at Paul Dakwins' excellent discussion of the method at

http://tutorial.math.lamar.edu/AllBrowsers/3401/ReductionofOrder.asp

Try this method for the ODE (2) and thus show that its general solution may indeed be written as $y(t) = (C + D\, t)\, e^{-k\, t}$, where C and D are constants.

[2] Well, your lecturer is obviously very very clever, but do you really think he's clever enough to simply have spotted this?