MATH10222: EXAMPLE SHEET[1] II

Questions for supervision classes

Hand in the solutions to questions 1, 2a, 3a, 4b and 5b. Attempt all other questions too and raise any problems with your supervisor.

1. **Existence, uniqueness and graphical solutions**
In this question we will explore the solutions of the initial value problem (IVP)

$$yy' = x - 1,$$

subject to the initial condition

$$y(x = X) = Y,$$

where X and Y are constants.

(a) For which values of X and Y does the existence and uniqueness theorem ensure the existence of a unique solution? If a unique solution can be shown to exist, can you be certain that it will exist for *all* values of x?

(b) Without solving the ODE, sketch its isoclines and integral curves. Does the ODE have a critical point? Does the sketch suggest the existence of any asymptotes, i.e. limiting curves that all solutions approach?

(c) Now find the *general* solution of the ODE, using any suitable method.

(d) Relate the integral curves sketched in part 1b to the solution obtained in part 1c. Examine the solution for particular values of the constant of integration in order to identify any asymptotes, and to analyse the behaviour of solutions in the vicinity of the critical point.

(e) Relate the sketch of the solutions to the existence and uniqueness results obtained in part 1a.

2. **Separable ODEs**
For each of the following differential equations, find the general solution and sketch the integral curves.

(a)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}$$

[This involves a standard integral that you should know – if you can't remember it, recall that integrals involving $\sqrt{1 + x^2}$ are best done using the substitution $x = \sinh z$, and the identity $\cosh^2 z - \sinh^2 z = 1$.]

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(b)

$$\frac{dy}{dx} = \frac{4x}{(1+x^2)^{1/3}}$$

[Easy.]

(c)

$$\frac{dy}{dx} = \frac{-2y}{x-2}$$

[Be careful not to overlook any solutions.]

(d)

$$\sqrt{1+x^2}\,\frac{dy}{dx} = y$$

[Again, be careful not to overlook any solutions. The solution involves the same integral as in 2a. Sketching the solution is easier if you write $\text{arcsinh}(x) + C$ as $\ln(x + \sqrt{1+x^2}) + \ln|K|$.]

3. **Initial value problems**
Using the results from question 2, solve the following initial value problems:

(a)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} \quad \text{subject to } y(0) = 5.$$

(b)

$$\sqrt{1+x^2}\,\frac{dy}{dx} = y \quad \text{subject to } y(0) = -3.$$

4. **First-order ODEs of homogeneous type**
Show that the following ODEs are of homogeneous type and find their general solutions:

(a)

$$x\,y\,\frac{dy}{dx} + x^2 + y^2 = 0.$$

(b)

$$x^2\,\frac{dy}{dx} + y^2 - xy = 0.$$

5. **First-order linear ODEs**
Solve the following initial value problems:

(a)

$$(1-x^2)\,\frac{dy}{dx} - x\,y = 1 \quad \text{subject to } y(0) = 0.$$

For what range of values of x is the solution is defined? Sketch the solution. [**Hint:** $\int (1-x^2)^{-1/2}\,dx = \arcsin x$, again a standard integral that you should know. The substitution $x = \sin z$ and the identity $\sin^2 x + \cos^2 x = 1$ does the trick...]

(b)

$$\frac{dy}{dx} - \frac{y}{x} = x\,\cos x \qquad \text{subject to } y(\pi) = 0.$$

Sketch the solution.