4 Mechanics applications of second-order ODEs

- Second-order linear ODEs with constant coefficients arise in many physical applications. One physical systems whose behaviour is governed by an ODE of the form

$$m\ddot{x} + k\dot{x} + cx = f(t) \tag{2}$$

is the damped mechanical oscillator shown in Fig. 1. In this case, m represents the mass of the particle attached to the spring, k is a measure of the strength of the damper, and c represents the spring stiffness. $f(t)$ is the applied external force. t represents time and $x(t)$ is the displacement of the mass from its rest position.

- The physical interpretation of the individual terms in equation (2) is as follows. The term $k\dot{x}$ represents the force exerted by the damper on the mass: for a linear damper, this force is proportional to the velocity $\dot{x}$ and it resists the motion. The term cx represents the force exerted by the spring on the mass: for a linear spring, this force is proportional to the displacement x and it acts in the direction opposite to the displacement. Equation (2) expresses Newton's law, which states that the sum of all forces acting on the particle is equal to its mass times its acceleration, $m\ddot{x}$.
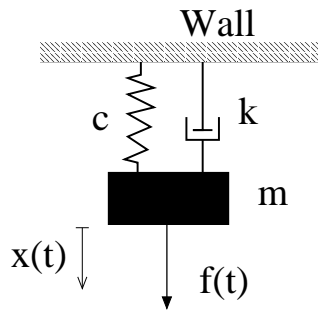

Figure 1: Sketch of a damped mechanical oscillator consisting of a mass m attached to a rigid wall by a linear spring of spring stiffness c and a damper with damping constant k. A time-dependent force, $f(t)$, is applied to the mass and the displacement of the mass from its rest position is represented by $x(t)$.

- The initial conditions are given by the initial position,

$$x(t = 0) = x_0,$$

and the initial velocity of the mass at time $t = 0$,

$$\left.\frac{dx}{dt}\right|_{t=0} = v_0.$$

4.1 The unforced case: $f(t) = 0 -$ Eigenfrequencies

- If $f(t) = 0$, equation (2) is reduced to its *homogeneous* form,

$$m\ddot{x} + k\dot{x} + cx = 0. \tag{3}$$

which describes the motion of the mass in absence of any external forcing.

- To reduce the number of parameters required to classificy the character of the particle's motion, we re-write the ODE (3) as

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = 0, \tag{4}$$

where

$$\delta = \frac{k}{2m} \quad \text{and} \quad \omega^2 = \frac{c}{m}. \tag{5}$$

- Note that in the damped mechanical oscillator the coefficients m, k and c are positive. Hence δ and ω^2 are positive, too.

- The solution to the corresponding characteristic equation

$$\lambda^2 + 2\delta\lambda + \omega^2 = 0, \tag{6}$$

  is given by

$$\lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega^2} \tag{7}$$

  or, written in a different form,

$$\lambda_{1,2} = -\delta \pm i\sqrt{\omega^2 - \delta^2}. \tag{8}$$

- This allows a straightforward identification of four distinct types of motion:

I. Purely damped motion: $\delta > \omega$

In this case equation (7) shows that both roots are real and the solution is given by

$$x(t) = Ae^{(-\delta + \sqrt{\delta^2 - \omega^2})t} + Be^{(-\delta - \sqrt{\delta^2 - \omega^2})t}. \tag{9}$$

Since both exponents are negative, this represents a *purely damped motion*, i.e. the system does not perform any oscillations.
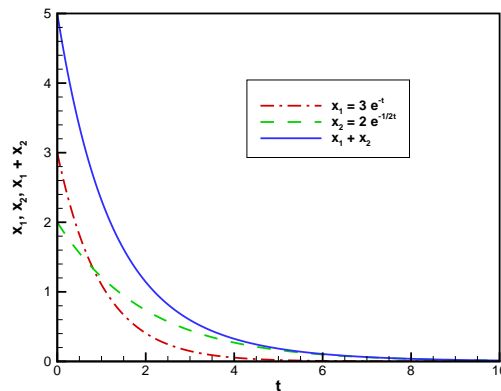

Figure 2: Illustration of a purely damped motion. The mass approaches its equilibrium position $x = 0$ monotonically.

II. Critically damped motion: $\delta = \omega$

In this case the square root in (7) vanishes and we have $\lambda_1 = \lambda_2 = -\delta$ and the general solution is given by

$$x(t) = Ae^{-\delta t} + Bte^{-\delta t}. \tag{10}$$

This represents a *critically damped motion* in which $x(t)$ approaches zero but can cross the value $x = 0$ (at most) once (when $t = -A/B$; it depends on the initial conditions which determine A and B, if this happens for $t > 0$).

III. Damped oscillation: $\delta < \omega$

In this case (8) shows that both roots are complex. The general solution is then given by

$$x(t) = e^{-\delta t}\left(A\cos(t\sqrt{\omega^2 - \delta^2}) + B\sin(t\sqrt{\omega^2 - \delta^2})\right). \tag{11}$$

This solution represents a *damped oscillation* with frequency $\sqrt{\omega^2 - \delta^2}$ whose amplitude decays exponentially.
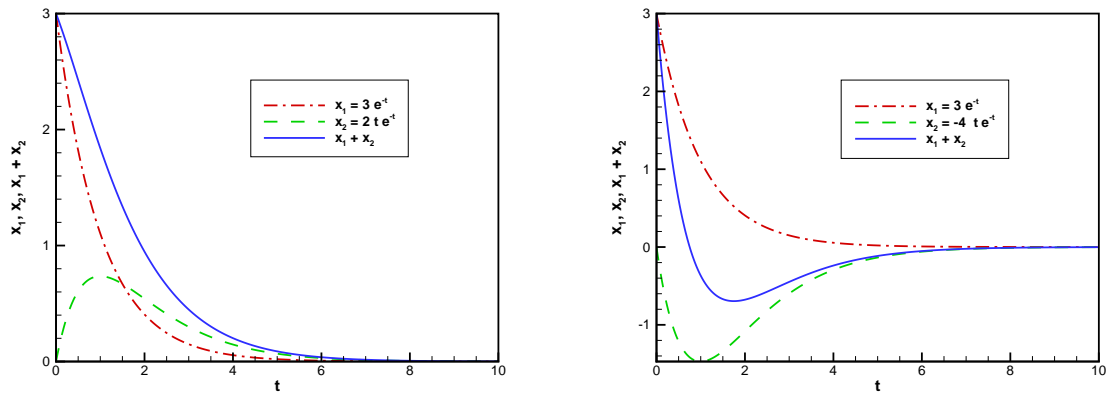
Figure 3: Illustration of critically damped motions. The mass approaches its equilibrium position, $x = 0$, with at most one "overshoot".
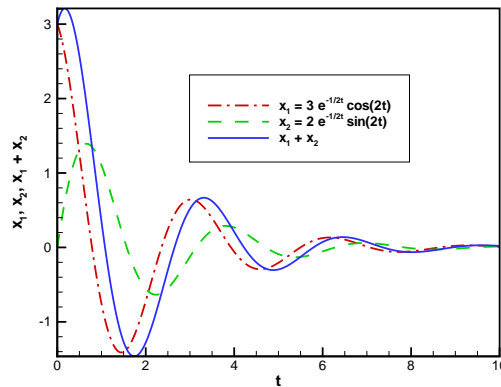

Figure 4: Illustration of a damped oscillation. The mass oscillates about its equilibrium position $x = 0$ and the amplitude of the oscillations decays exponentially.

IV: Undamped oscillations $\delta = 0$

The solution (11) is still valid and for $\delta = 0$ we obtain

$$x(t) = A\cos(\omega t) + B\sin(\omega t), \tag{12}$$

which is an *undamped oscillatory motion* with *eigenfrequency* ω.

4.2   The inhomogeneous equation – Periodic forcing and resonance

- The general solution of the inhomogeneous equation

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = F(t), \tag{13}$$

where $F(t) = f(t)/m$ is given by

$$x(t) = x_H(t) + x_P(t). \tag{14}$$

- In (14), $x_H(t)$ is the general solution of the corresponding homogeneous equation (4) and thus contains the two free constants required to fulfill the initial conditions. $x_P(t)$ (the particular solution) is *any* solution of the inhomogeneous equation.
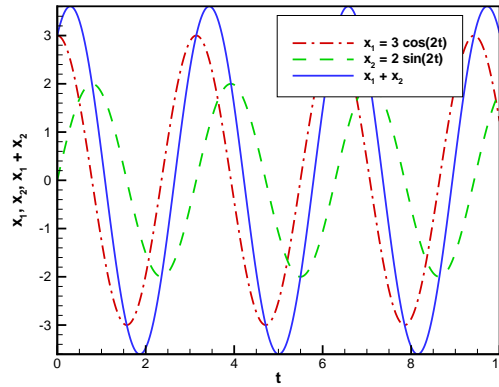
Figure 5: Illustration of an undamped oscillation. The mass performs harmonic oscillations about its equilibrium position $x = 0$.

- The most important form of forcing in many engineering applications is the harmonic forcing in which $F(t)$ has the form

$$F(t) = F\sin(\Omega t) \quad \text{or} \quad F(t) = F\cos(\Omega t).$$

- It is advantageous to carry out the calculation with complex numbers by writing

$$F(t) = Fe^{i\Omega t} = F\cos(\Omega t) + iF\sin(\Omega t). \tag{15}$$

Once the calculation is completed we can extract the relevant real solution (corresponding to the cos or sin forcing) by taking the real or imaginary parts of the complex solution.

- We know from the previous section that we can find a solution for exponential forcing in the form

$$x_P(t) = Xe^{i\Omega t}. \tag{16}$$

Hence we insert (16) and (15) into (13) and carry out the differentiations.

- After cancelling the common factor $e^{i\Omega t}$, this provides the following equation for the unknown amplitude X (which is complex in general):

$$X = \frac{F}{(\omega^2 - \Omega^2) + i\,(2\delta\Omega)} = F\,\frac{(\omega^2 - \Omega^2) - i\,(2\delta\Omega)}{(\omega^2 - \Omega^2)^2 + (2\delta\Omega)^2} \tag{17}$$

- Now we can multiply out $x_P(t) = Xe^{i\Omega t} = X(\cos(\Omega t) + i\sin(\Omega t))$ and extract the real or imaginary part to obtain the relevant real solution, i.e.

$$x_P(t) = \mathcal{R}(X)\cos(\Omega t) - \mathcal{I}(X)\sin(\Omega t) \qquad \text{for} \quad F(t) = F\cos(\Omega t)$$

and

$$x_P(t) = \mathcal{R}(X)\sin(\Omega t) + \mathcal{I}(X)\cos(\Omega t) \qquad \text{for} \quad F(t) = F\sin(\Omega t).$$

- Alternatively, we can re-write (17) in polar form and thus obtain the *amplitude* of the response

$$|X| = \frac{F}{\sqrt{(\omega^2 - \Omega^2)^2 + (2\delta\Omega)^2}} = \frac{F/\omega^2}{\sqrt{\left(1 - (\Omega/\omega)^2\right)^2 + \left(2\,(\delta/\omega)\,(\Omega/\omega)\right)^2}}$$

and the *phase angle*

$$\varphi = \arg(X) = \arctan\left(\frac{-2\delta\Omega}{\omega^2 - \Omega^2}\right). \tag{18}$$

- Hence, the particular solution can also be written as

$$x_P(t) = |X| \, e^{i(\Omega t + \varphi)}.$$

The relevant real solutions are again obtained by taking the real and imaginary part of this complex solution, i.e.

$$x_P(t) = |X| \cos(\Omega t + \varphi) \qquad \text{for} \quad F(t) = F \cos(\Omega t)$$

and

$$x_P(t) = |X| \sin(\Omega t + \varphi) \qquad \text{for} \quad F(t) = F \sin(\Omega t).$$

Comparing this to (15) shows that the particular solution (i.e. the forced motion), $x_P(t)$, has a phase difference of φ against the forcing, $F(t)$.

- Note that for weak damping (small δ), the amplitude of the response, $|X|$, becomes very large when the excitation frequency Ω is close to the eigenfrequency of the system, $\Omega \approx \omega$. This is known as *resonance*. For vanishing damping, $\delta = 0$, the response becomes unbounded when $\Omega \to \omega$. This is the 'resonance catastrophe'.

In practical applications there is always some damping but the amplitudes of the forced oscillations of many physical systems can still become so large that the system is destroyed when the excitation frequency is sufficiently close to the eigenfrequency of the system. [Remember the wobbling Millenium Bridge?]

Here is a plot of the (normalised) amplitude of the oscillation as a function of the excitation frequency (normalised by the system's eigenfrequency ω) for various values of the damping parameter δ:

Normalised amplitude of the oscillation of the harmonically forced mechanical oscillator
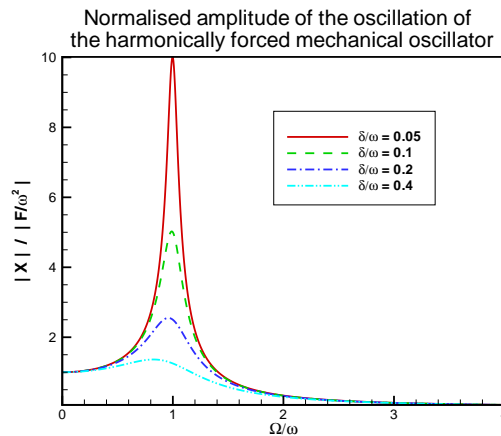
- Finally, note that for positive damping ($\delta > 0$), the homogeneous solution $x_H(t)$ decays rapidly with increasing time, i.e. $x_H(t) \to 0$. For sufficiently large times, only the forced motion $x_P(t)$ persists. Therefore, the solution $x_H(t)$ is often referred to as the *transient solution*.
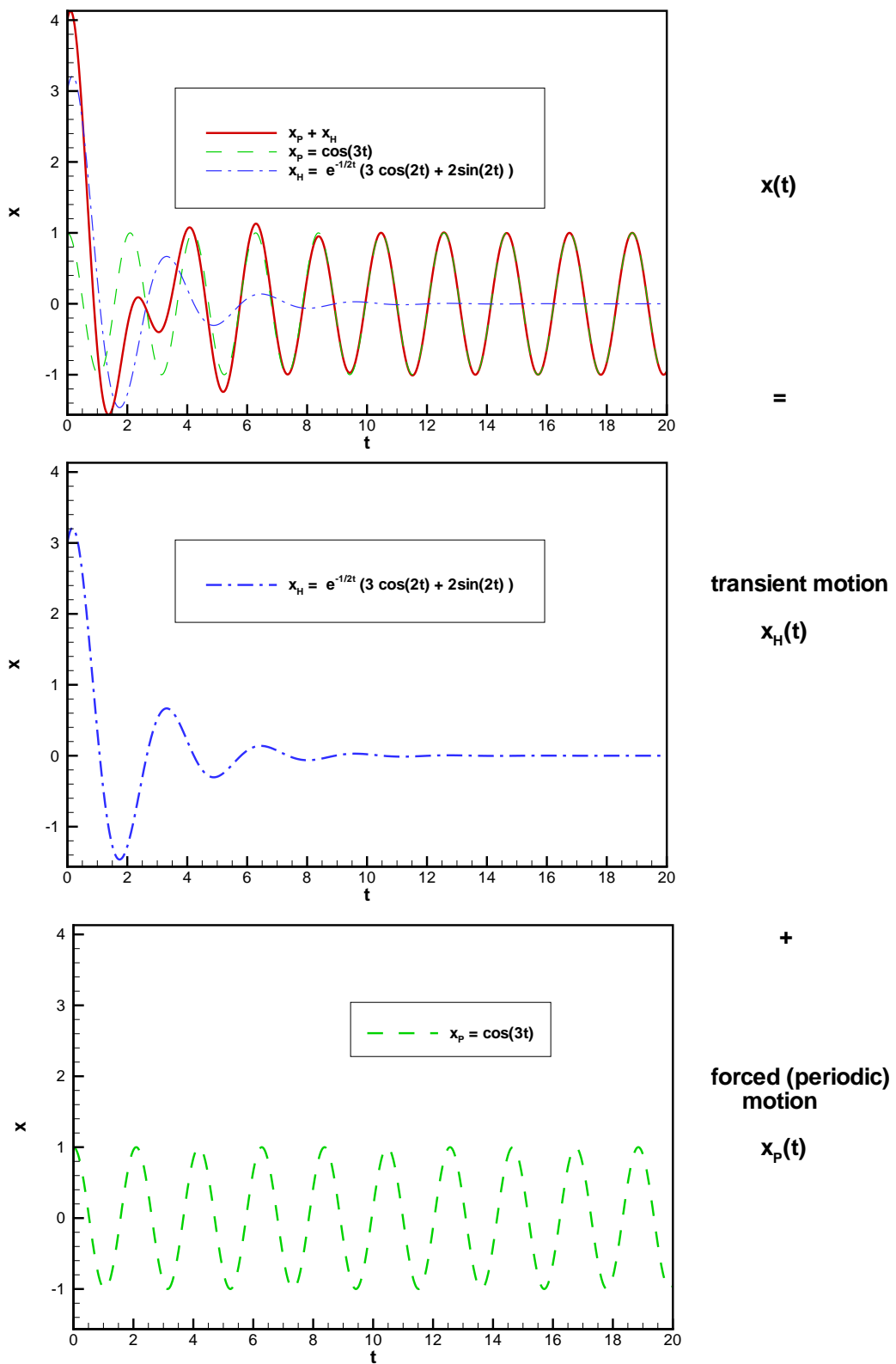
Figure 6: The displacement of a harmonically-forced, damped mechanical oscillator comprises the periodic (forced) solution $x_P(t)$ and the transient solution $x_H(t)$.