3 Second-Order Ordinary Differential Equations

The general form of a second-order ODE is given by

$$\mathcal{F}\big(x, y(x), y'(x), y''(x)\big) = 0.$$

It is typically augmented by two boundary or initial conditions, i.e constraints of the form

$$y(X) = Y, \quad y'(X) = Z,$$

or

$$y(X_1) = Y_1, \quad y(X_2) = Y_2$$

where the constants X, Y, Z (or X_1, Y_1, X_2, Y_2) are given.

Often the ODE can be written in explicit form as

$$y''(x) = f\big(x, y(x), y'(x)\big).$$

In this lecture we will mainly concentrate on linear second-order ODEs. (In section 3.3 we will briefly discuss the solution of two particular types of nonlinear ODEs).

3.1 Some theory for linear second-order ODEs

- In general, we shall write a *linear* second-order ODE for $y(x)$ in one of two ways, either as

  $$a(x)\, y'' + b(x)\, y' + c(x)\, y = d(x)$$

  or as

  $$y'' + p(x)\, y' + q\,(x) y = r(x).$$

  We will take these ODEs to be defined on an interval

  $$I = (\alpha, \beta) = \{\, x \mid \alpha < x < \beta \,\}$$

  which is chosen such that, at all values of x in I:

  - $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are defined and continuous
  - and $a(x)$ is never zero

  so that the functions $p(x)$, $q(x)$ and $r(x)$, which are defined as

  $$p(x) = b(x)/a(x), \quad q(x) = c(x)/a(x), \quad r(x) = d(x)/a(x),$$

  are also defined and continuous throughout I.

- **Theorem (*Existence and Uniqueness*):** If $y(x)$ satisfies the ODE $y'' + p(x)\, y' + q(x)\, y = r(x)$, and the functions $p(x)$, $q(x)$ and $r(x)$ are continuous throughout the interval I, then there is only one solution that satisfies the pair of initial conditions

  $$y(X) = Y \quad \text{and} \quad y'(X) = Z$$

  and this solution exists throughout the interval I.

  This theorem guarantees that solutions will exist throughout the interval I and that the two initial conditions, one giving the value of y and the other giving the value of its derivative y', both specified at the same point in I, are enough to select a unique solution.

  Note that the existence and uniqueness theorem only applies to initial value problems!

- **Superposition:** In the special case in which the ODE has $r(x)$ set equal to zero, that is for the special form of the ODE

  $$y'' + p(x)\, y' + q(x)\, y = 0$$

  which is known as the 'homogeneous' form of the ODE, a linear combination of any solutions is also a solution. Thus if $y_1(x)$ and $y_2(x)$ are solutions of $y'' + p(x)\, y' + q(x)\, y = 0$ then so is any function that can be written as $A\, y_1(x) + B\, y_2(x)$ for any constants A and B.

- **Fundamental Solutions:** What is more, *any solution* of the homogeneous second-order linear ODE $y'' + p(x)\,y' + q(x)\,y = 0$ can be written as a linear combination of only two solutions $y_1(x)$ and $y_2(x)$, known as 'fundamental solutions,' provided $y_1(x)$ and $y_2(x)$ are nonzero and linearly independent.

  [Reminder: Two functions $y_1(x)$ and $y_2(x)$, defined on I, are said to be ***linearly independent*** on I if the only linear combination of them that adds up to zero, so that $A\,y_1(x) + B\,y_2(x) = 0$ for all $x \in I$, is the one for which $A = B = 0$.]

  The choice of fundamental solutions is not unique. For instance, if $\{y_1(x), y_2(x)\}$ is a set of fundamental solutions for a given linear homogeneous ODE then $\big\{y_1(x), \big(y_1(x) + y_2(x)\big)\big\}$ is another set of fundamental solutions.

  A solution of the homogeneous ODE is sometimes called a *complementary function.*

- **General Solutions:** Any solution of the non-homogeneous ODE $y'' + p(x)\,y' + q(x)\,y = r(x)$ has the form, known as the 'general solution'

$$y = y_P(x) + A\,y_1(x) + B\,y_2(x)$$

  where $y_P(x)$, known as a 'particular solution,' is a solution of the non-homogeneous ODE, and $y_1(x)$ and $y_2(x)$ are fundamental solutions of the homogeneous form of the ODE, in which $r(x)$ is set to zero.

- The solution to a specific boundary or initial value problem can therefore be obtained in four steps:

  1. Find the general solutions of the homogeneous ODE:

  $$y'' + p(x)\,y' + q(x)\,y = 0 \quad \Longrightarrow \quad y_H(t) = A\,y_1(t) + B\,y_2(t),$$

     where $y_1(t)$ and $y_2(t)$ are two nonzero, linearly independent solutions, i.e. they are fundamental solutions of the homogenous ODE.

  2. Find a particular solution of the inhomogeneous ODE

  $$y'' + p(x)\,y' + q(x)\,y = r(x) \quad \Longrightarrow \quad y_P(t).$$

  3. Write down the general solution

  $$y(x) = y_P(x) + y_H(t) = y_P(t) + A\,y_1(x) + B\,y_2(x).$$

  4. Determine the constants A and B from the boundary or initial conditions.

3.2 Linear second-order ODEs with constant coefficients

3.2.1 The general solution of the homogenous ODE

- Second-order ODEs for $y(x)$ of the form

$$y'' + py' + qy = 0 \quad \text{with} \;\; p \;\; \text{and} \;\; q \;\; \text{constant}$$

  can always be solved, for all real values of x, using the ansatz

$$y = e^{\lambda x}.$$

  [***Important*:** The method does not generally work when p and q are not constant.]

- Inserting $y = e^{\lambda x}$ into the ODE and cancelling the common factor $e^{\lambda x}$ yields the so-called *characteristic polynomial*

$$\lambda^2 + p\lambda + q = 0 \quad \text{with roots} \quad \lambda = \tfrac{1}{2}\Big(-p \pm \sqrt{p^2 - 4q}\Big).$$

  The roots, and hence the nature of the solutions, depends on the sign of the 'discriminant' $p^2 - 4q$:

Case 1: $p^2 - 4q > 0$

If the discriminant is positive ($p^2 - 4q > 0$) then λ has two distinct real roots of the form

$$\lambda_1 = \tfrac{1}{2}\Big(-p - \sqrt{p^2 - 4q}\Big) \quad \text{and} \quad \lambda_2 = \tfrac{1}{2}\Big(-p + \sqrt{p^2 - 4q}\Big).$$

The general solution of the homogenous ODE can therefore be written as

$$y = A\, \mathrm{e}^{\lambda_1 x} + B\, \mathrm{e}^{\lambda_2 x},$$

where A and B are arbitrary constants.

Case 2: $p^2 - 4q < 0$

If the discriminant is negative ($p^2 - 4q < 0$) then λ has two complex conjugate roots of the form

$$\lambda = \mu \pm i\omega \quad \text{with} \quad \mu = -\tfrac{1}{2}p \quad \text{and} \quad \omega = \tfrac{1}{2}\sqrt{4q - p^2}.$$

The general solution of the homogeneous ODE can then be written as

$$y = A\, \mathrm{e}^{\mu x}\cos(\omega x) + B\, \mathrm{e}^{\mu x}\sin(\omega x),$$

where A and B are arbitrary constants.

Case 3: $p^2 - 4q = 0$

If the discriminant is zero ($p^2 - 4q = 0$) then the characteristic polynomial has one double root

$$\lambda_{1,2} = \lambda = -\tfrac{1}{2}p$$

giving only one fundamental solution $y_1 = \mathrm{e}^{\lambda x} = \mathrm{e}^{-px/2}$. However another fundamental solution is $y_2 = x\,\mathrm{e}^{\lambda x} = x\,\mathrm{e}^{-px/2}$ (*Exercise*: check this by substitution). The general solution of the homogeneous ODE can therefore be written as

$$y = A\, \mathrm{e}^{-px/2} + B\, x\,\mathrm{e}^{-px/2},$$

where A and B are arbitrary constants.

3.2.2 The particular solution of the inhomogenous ODE: The method of undetermined coefficients

- The method of undetermined coefficients is, more or less, a process of trial and error, or guesswork, based on making a suitable initial assumption about the overall form of the solution.
- The method and its pitfalls are best illustrated with an example:

$$y'' + py' + qy = A\,\mathrm{e}^{ax}.$$

Initial ansatz:

Given that the RHS e^{ax} retains its functional form when differentiated, it is tempting to try a solution in the form $y = C\,\mathrm{e}^{ax}$, having $y' = Ca\,\mathrm{e}^{ax}$ and $y'' = Ca^2\,\mathrm{e}^{ax}$, so that

$$Ca^2\,\mathrm{e}^{ax} + pCa\,\mathrm{e}^{ax} + qC\,\mathrm{e}^{ax} = A\,\mathrm{e}^{ax} \quad \text{or} \quad (a^2 + pa + q)C = A$$

which requires that $C = \frac{A}{a^2+pa+q}$, leading to the particular solution

$$y = y_{\mathrm{p}}(x) = \frac{A}{a^2 + pa + q}\,\mathrm{e}^{ax} \quad \text{provided} \quad a^2 + pa + q \neq 0.$$

Modification if a is a (single) root of the characteristic polynomial

If $a^2 + pa + q = 0$ the initial ansatz, that $y = C\,\mathrm{e}^{ax}$, is obviously inadequate. We note that this case arises if the a happens to be a root of the characteristic polynomial of the associated homogeneous ODE. In this case, another ansatz is appropriate. We assume, instead, that

$$y = Cx\,\mathrm{e}^{ax} \quad \text{so that} \quad y' = C(1 + ax)\,\mathrm{e}^{ax}, \quad y'' = C(2a + a^2x)\,\mathrm{e}^{ax}.$$

In this case the ODE gives

$$C(2a + a^2 x)\,\mathrm{e}^{ax} + pC(1 + ax)\,\mathrm{e}^{ax} + qCx\,\mathrm{e}^{ax} = A\,\mathrm{e}^{ax}$$

or

$$\big(x\,\underbrace{(a^2 + ap + q)}_{=0} + 2a + p\big)C = (2a + p)C = A$$

since $a^2 + ap + q = 0$. Thus we find that $C = \frac{A}{2a+p}$, leading to the particular solution

$$y = y_\mathrm{p}(x) = \frac{A}{2a+p}\,x\,\mathrm{e}^{ax} \quad \text{provided} \quad a^2 + pa + q = 0 \quad \text{and} \quad 2a + p \neq 0.$$

Modification if a is a double root of the characteristic polynomial

If both $a^2 + pa + q$ and $2a + p$ are zero, then both guesses, that $y = C\,\mathrm{e}^{ax}$ or $y = Cx\,\mathrm{e}^{ax}$, are obviously inadequate. We note that this case arises if a is a double root of the characteristic polynomial. In this case, yet another ansatz is appropriate. We now assume that

$$y = Cx^2\,\mathrm{e}^{ax} \quad \text{so that} \quad y' = C(2x + ax^2)\,\mathrm{e}^{ax}, \quad y'' = C(2 + 4ax + a^2x^2)\,\mathrm{e}^{ax}.$$

In this case the ODE gives

$$C(2 + 4ax + a^2x^2)\,\mathrm{e}^{ax} + pC(2x + ax^2)\,\mathrm{e}^{ax} + qCx^2\,\mathrm{e}^{ax} = A\,\mathrm{e}^{ax}$$

or

$$\big(x^2\,\underbrace{(a^2 + ap + q)}_{=0} + x\,\underbrace{(2a + p)}_{=0} + 2\big)C = 2C = A$$

since $a^2 + ap + q = 0$ and $2a + p = 0$. Thus we find that $C = \frac{1}{2}A$, leading to the particular solution

$$y = y_\mathrm{p}(x) = \tfrac{1}{2}A\,x^2\,\mathrm{e}^{ax} \quad \text{provided} \quad a^2 + pa + q = 0 \quad \text{and} \quad 2a + p = 0.$$

- This example shows that a particular solution of the ODE $y'' + py' + qy = A\,\mathrm{e}^{ax}$, with constant coefficients p and q, typically takes the form $Cx^m\,\mathrm{e}^{ax}$ for an integer power m that depends on whether or not e^{ax} and $x\,\mathrm{e}^{ax}$ are solutions of the homogeneous equation.
- Based on this observation we can formulate the "method of undetermined coefficients" for inhomogenous, constant-coefficient of the form

$$y'' + p\,y' + q\,y = A_1\,r_1(x) + A_2\,r_2(x) + \cdots + A_n\,r_n(x)$$

where the RHS is a linear combination of n given, linearly-independent functions $r_i(x)$ $(i = 1, ..., n)$.

The idea is the following:

1. We initially try to find a particular solution that contains the same (linearly independent) functions that occur on the RHS:

$$y_P^{[initial]}(x) = C_1\,r_1(x) + C_2\,r_2(x) + \cdots + C_n\,r_n(x)$$

with undetermined (constant) coefficients C_i $(i = 1, ..., n)$. The plan is to insert this into the ODE and to collect the coefficients that multiply the same functions $r_i(x)$ $(i = 1, ..., n)$. Since the $r_i(x)$ are linearly independent, their linear combination can only vanish if the coefficients multiplying them vanish individually. This provides n equations for the n unknown coefficients C_i $(i = 1, ..., n)$. Bingo!

2. This doesn't work, however, if the derivative of any of the $r_i(x)$ cannot be expressed as a linear combination of the terms in $y_P^{[initial]}$. [In the above example, the derivatives of $r_1(x) = e^{ax}$ were simply multiples of e^{ax}, so no additional functions arose. However, if $r_1(x) = x^2$, say, the differentiation of $y_P^{[initial]}$ would also produce $r_1'(x) = 2x$ and $r_1''(x) = 2$.]
To deal with such cases, we generalise our ansatz to the form

$$\begin{aligned} y_P^{[better]}(x) \;=\;& C_1\,r_1(x) + C_2\,r_2(x) + \cdots + C_n\,r_n(x) \\ +\;& D_1\,r_1'(x) + D_2\,r_2'(x) + \cdots + D_n\,r_n'(x) \\ +\;& E_1\,r_1''(x) + E_2\,r_2''(x) + \cdots + E_n\,r_n''(x), \end{aligned}$$

where we set the coefficients E_i and D_i $(i = 1, ..., n)$ that multiply terms that are already contained in $y_P^{[initial]}(x)$ to zero.

3. Finally, we have to deal with the case where some of the terms in $y_P^{[better]}$ are solutions of the homogenous ODE $y'' + p\,y' + q\,y = 0$. Let $\tilde{r}(x)$ be a term in $y_P^{[better]}(x)$ that is a solution of the homogeneous ODE. For each such term, we replace $\tilde{r}(x)$ by $x^m\,\tilde{r}(x)$, where m is the smallest positive integer for which $x^m\,\tilde{r}(x)$ does not solve the homogeneous ODE. If the derivatives of $x^m\,\tilde{r}(x)$ create new linearly independent functions, not yet contained in $y_P^{[better]}$, add these too.

3.3 Some nonlinear second-order ODEs

In a few cases, second-order ODEs can be solved as first-order ODEs. Two important cases are those that take the form

$$\tfrac{\mathrm{d}^2y}{\mathrm{d}t^2} = f\big(y, \tfrac{\mathrm{d}y}{\mathrm{d}t}\big) \qquad \text{or} \qquad \tfrac{\mathrm{d}^2y}{\mathrm{d}t^2} = f\big(t, \tfrac{\mathrm{d}y}{\mathrm{d}t}\big)$$

when describing $y(t)$. The first of these represents second-order ODEs that are autonomous, which is to say that they do not depend on t (apart from differentiating with respect to t). The second represents second-order ODEs that do not depend on y (except as derivatives of y).

3.3.1 Second-order ODEs for $y(t)$ that do not depend on y

Such ODEs take the form

$$y'' = f(t, y').$$

All we need to do is note that this is actually a first-order ODE for $y'(t)$. If we write, $v(t) = y'(t)$ then the ODE is clearly a first-order ODE for v, namely

$$v' = f\big(t, v\big).$$

If this is solved to find a solution $v(t)$, then $y(t)$ is a solution of the first-order ODE $y' = v(t)$.

3.3.2 Autonomous second-order ODEs

Autonomous second-order ODEs which, when describing $y(t)$ have the form

$$y'' = f(y, y')$$

can also be solved by writing $v = y'(t)$, but in a different way. Differentiating $y'(t) = v$ gives

$$y'' = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}y}.$$

The ODE can therefore be rewritten in the form

$$v\tfrac{\mathrm{d}v}{\mathrm{d}y} = f(y, v)$$

which, if we think of v as being a function of y, is a first-order ODE for v. If we can solve for $v = v(y)$ then $y(t)$ is a solution of the first-order ODE $y' = v(y)$.