2 First-Order Ordinary Differential Equations

First-order ordinary differential equations describing $y(x)$ have the forms

$$F(x, y, y') = 0 \qquad \text{or} \qquad y' = f(x, y).$$

The first of these forms is more general, but where it can be solved for y', it can be written in the second form. In this section we shall consider a variety of ways in which solutions can be obtained for particular forms of first-order ODE.

2.1 Graphical approach

Before seeking any actual solutions, however, it can be noted that the ODE itself contains a lot of information about the nature of its solutions. This is because the equation

$$y' = f(x, y)$$

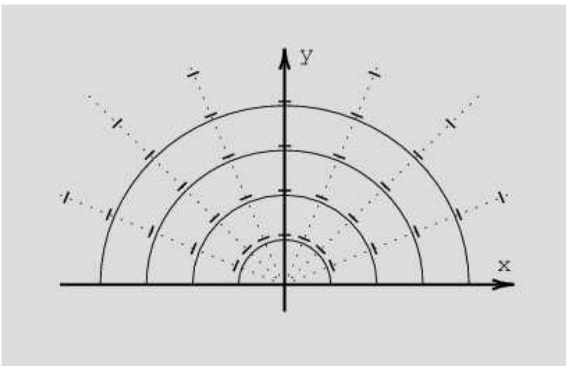
gives the slope of the function $y(x)$. In the plane of x and y it provides the direction in which the solution must be changing at any point (x, y). This is called the direction field of the ODE.

- **The direction field** of the ODE $y' = f(x, y)$ is the set of all direction vectors having the same direction as the vector $(1, y')$, at each point (x, y), in the plane of x and y.
- **Integral curves** are curves which are everywhere tangent to the direction field. Each integral curve represents a solution of the ODE.
- **Example 1:** If $y' = -x/y$, for $y(x) > 0$, then at any point (x, y) the direction in which the solution changes is $(1, -x/y)$.

  The direction does not change if it is multiplied by a scalar, so this is the same as the direction $(y, -x)$.

  At a point (x, y), the direction of $(y, -x)$ is at right angles to the line from the origin to (x, y), since $(x, y) \cdot (y, -x) = 0$. We can therefore easily sketch the direction field as short vectors at right angles to any line emerging from the origin.

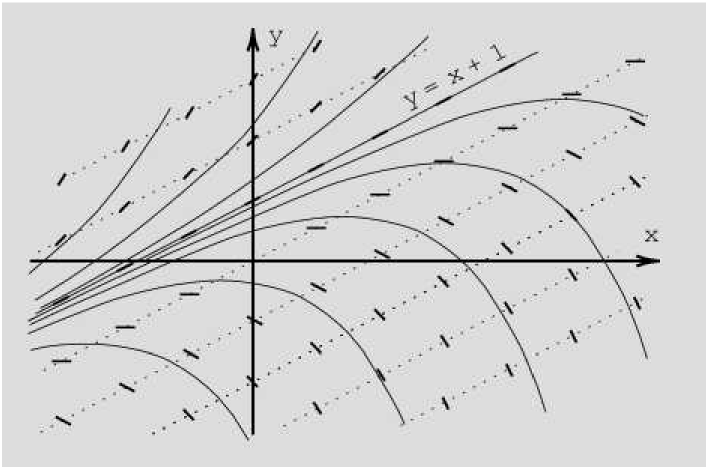
  It is clear that the integral curves in this example, which are tangent to the direction field at any point, must be semicircles in the half-plane $y > 0$, as indicated by the thin lines in the illustration.



A note on Existence and Uniqueness: Only one circle, with a fixed centre, can pass through any given point. Thus, if we specify some point, via a initial condition at which $y(\bar{x}) = \bar{y} > 0$, that must lie on a solution of $y' = -x/y$, this picks out a unique semicircle, with its centre at $(0, 0)$, as the solution that passes through $(\bar{x}, \bar{y})$.

More generally, for any first-order ODE $y' = f(x, y)$, if $f(x, y)$ and $f_y(x, y)$ are continuous, the direction field does not change direction abruptly. Intuitively, since there is only one direction in which the path can move at any point, there can be only one path, starting at some point $(\bar{x}, \bar{y})$, that is tangent to all of these smoothly changing direction vectors. It is then clear that only one integral curve (or solution) can pass through any given point $(\bar{x}, \bar{y})$.

- Direction fields are not always as easy to identify as in the example above. More generally, it is useful to identify isoclines and critical points as a means of building up a picture of the direction field:
  - **Isoclines** are paths in the space of x and y on which $y' = const.$ Clearly, the directions in the direction field are the same along any one isocline.
  - **Critical points** arise where isoclines with different directions intersect.
  - **Example:** Considering the previous example $y' = -x/y$ again, in which each of the radii is an isocline, the origin $x = y = 0$ is a critical point.
- **Example 2:** If we consider $y' = y - x$, isoclines are given by the straight lines $y = x + c$, for a constant c, along which $y' = c$. The corresponding direction vector is thus $(1, c)$. Knowing this, the direction field and integral curves can be sketched fairly easily:



Note that the path $y = x + 1$ has all of its direction vectors pointing in the same direction as the path, so that $y = x + 1$ is also an integral curve (a solution). It identifies a natural asymptote for other integral curves.

There are no critical points for this ODE.

- The graphical approach helps to uncover qualitative behaviour of solutions of ordinary differential equations. Where quantitative results are needed, solutions must be found.