MATH10222 Lecture Notes

This set of notes summarises the main results of the first half of the lecture MATH10222 (Calculus and Applications). Please email any corrections (yes, there might be the odd typo...) or suggestions for improvement to *M.Heil@maths.man.ac.uk* or see me after the lecture.

Generally, the notes will be handed out after the material has been covered in the lecture. You can also download them from the WWW:

http://www.maths.man.ac.uk/˜mheil/Lectures/FirstYearODEs/.

This WWW page will also contain announcements, example sheets, solutions, etc.

This course does not follow any particular textbook – your lecture notes and the handouts will be completely sufficient. If you bought Stewart's textbook for the first-semester courses, you can, of course, consult it on any of the topics covered in this lecture.

If you want a concise overview of the theory plus lots and lots of worked examples, have a look at Richard Bronson's "Differential Equations" in the Schaum's Outline Series.

Finally, I can recommend Paul Dawkins' website

http://tutorial.math.lamar.edu/AllBrowsers/3401/

You may also wish to have a look at his "How to Study Math[s]" guide at

http://tutorial.math.lamar.edu/StudyMath/HowToStudyMath.asp

1 Generalities

1.1 Ordinary derivatives

- A differentiable function $y(x)$ of one independent variable, or "argument" x, has the derivative

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x) \quad \text{or} \quad y'(x)$$

(either notation meaning the same thing) which represents the rate at which $y(x)$ changes as x changes, at the point x. That is, $y'(x)$ is a function defined by the limit

$$y'(x) = \lim_{|h|\to 0} \frac{y(x+h) - y(x)}{h}.$$

- The function $y(x)$ is said to be differentiable in an interval $I \subseteq \mathbb{R}$ if this limit exists at all values of x in the interval.
- If $y'(x)$ is also differentiable, then a second derivative, $y''(x)$ or $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(x)$, can be defined by replacing y with y' in the definition. Similarly, provided it exists, a derivative of *order* n

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n}(x) \quad \text{or} \quad y^{(n)}(x)$$

can be defined by repeating the differentiation n times.

- **Alternative notations:** A superscript dot or a capital D are also sometimes used to denote differentiation. A subscript (normally signifying partial differentiation) can also be used to denote ordinary differentiation. Thus derivatives of $u(t)$ can be represented by[1]

$$\dot{u}(t) = \mathrm{D}\,u(t) = u_t(t) = u^{(1)}(t) = \frac{\mathrm{d}u}{\mathrm{d}t}(t) = u'(t), \qquad \ddot{u} = \mathrm{D}^2 u = u_{tt} = u^{(2)} = \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = u''.$$

- Note that it is not necessary to write out the argument, in this case '(t)', after each of the derivatives, if it is clearly understood that u, u', u'', etc. are all functions of t.

1.2 Ordinary Differential Equations

- **Ordinary Differential Equations:** An ordinary differential equation, or ODE, relates a function $y(x)$ to x and some of its derivatives, $y'(x)$, $y''(x)$, ..., $y^{(n)}(x)$. In general it has the form

$$F\big(x, y, y^{(1)}, y^{(2)}, \ldots, y^{(n)}\big) = 0$$

although we will often assume that it can be written in the form

$$y^{(n)} = f\big(x, y, y^{(1)}, y^{(2)}, \ldots, y^{(n-1)}\big).$$

Much of this course will be devoted to studying ordinary differential equations of this type. The *order* of an ordinary differential equation is the order of the highest derivative appearing in the equation.

- **Solutions:** A solution of the ODE $F\big(x, y, y^{(1)}, \ldots, y^{(n)}\big) = 0$ on an interval I is any function $\phi(x)$ such that ϕ and all of the derivatives $\phi^{(1)}$, $\phi^{(2)}$, ..., $\phi^{(n)}$ exist on I and

$$F\big(x, \phi, \phi^{(1)}, \ldots, \phi^{(n)}\big) = 0 \quad \text{for all} \quad x \in I.$$

- **Linear ODEs:** Linear ODEs have a rich theoretical foundation and they are simpler to analyse than nonlinear ODEs. The ordinary differential equation for $y(x)$

$$F\big(x, y, y^{(1)}, y^{(2)}, \ldots, y^{(n)}\big) = 0$$

[1] Yet other notations for derivatives are sometimes encountered in various texts.

is linear if $F\big(x, y, y^{(1)}, \dots, y^{(n)}\big)$ is linear in y and all derivatives of y (namely, all of the arguments y, $y^{(1)}$, $y^{(2)}$, ..., $y^{(n)}$). In other words, it is linear if the ODE can be written in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y - g(x) = 0$$

in which all of the coefficients g, a_0, a_1, ..., a_n depend only on x (that is, they do not depend on y or any derivatives of y).

- **Autonomous ODEs:** An ODE for $y(x)$ of the form $F\big(y, y^{(1)}, \dots, y^{(n)}\big) = 0$ in which the independent variable x does not appear, is said to be autonomous.
- **Examples:** The ODE $(1-t)u'' - tu = 0$ for $u(t)$ is non-autonomous and linear; the ODE $v'(z) + v'^2(z) - v(z) = 0$ for $v(z)$ is autonomous and nonlinear.

1.3 Some basic preliminaries

We will now look at a number of simple examples and basic features of ordinary differential equations, the use of additional information at particular points, and the way in which solutions of ODEs depend on such data.

1.3.1 Existence of solutions

- It is not always obvious that solutions will exist at all. For example, the following ODE for $y(x)$

$$y' + 1/y' = 0$$

cannot have a real-valued solution. The value of the derivative of any solution would have to be either $y' = i$ or $y' = -i$, where $i = \sqrt{-1}$, which is not real. Therefore, if we are dealing only with real functions, then there is no solution.

1.3.2 Non-uniqueness

- If we solve the very simple ODE $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, for $y(x)$, by integrating successively, we obtain

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = A_1 \quad \text{and} \quad y = A_1x + A_2$$

where A_1 and A_2 are "arbitrary" constants of integration. That is, the function $A_1x + A_2$ provides a solution of $y'' = 0$ whatever constant values are chosen for A_1 and A_2.

In fact, every solution of this ODE has the form $A_1x + A_2$ for constant values of A_1 and A_2. The solution is not unique since a different choice of values for A_1 and A_2 provides a different solution.

1.3.3 Boundary and initial conditions

- A unique solution can only therefore arise if there are additional constraints on the allowed values of the solution. In general, for an n^{th} order ODE, there must be n independent constraints, if there is to be a unique solution. These constraints are usually provided by "boundary conditions" or "initial conditions".
- **Example 1:** As has been seen, the simple ordinary differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ has the solution $y = A_1x + A_2$ for arbitrary values of A_1 and A_2. If we impose the two constraints

$$y(0) = 1 \quad \text{and} \quad y(1) = 0$$

then we find that:

$$\left.\begin{array}{ll} y(0) = 1: & A_1 \times 0 + A_2 = 1 \\ y(1) = 0: & A_1 \times 1 + A_2 = 0 \end{array}\right\} \quad \Longrightarrow \quad A_1 = -1 \quad \text{and} \quad A_2 = 1.$$

There is, therefore, only one solution, namely $y = -x + 1$, that satisfies the two constraints $y(0) = 1$ and $y(1) = 0$.

- **Example 2:** The ordinary differential equation $xy'' - (1+x)y' + y = 0$ has the solution $y = B_1 \mathrm{e}^x + B_2(1+x)$ for arbitrary values of B_1 and B_2. If we impose the constraints

$$y(1) = -1 \quad \text{and} \quad y'(1) = 0$$

then we find that:

$$\left.\begin{array}{ll} y(1) = -1 : & B_1 \mathrm{e} + 2B_2 = -1 \\ y'(1) = 0 \;\; : & B_1 \mathrm{e} + \;\, B_2 \;\, = 0 \end{array}\right\} \quad \Longrightarrow \quad B_1 = \mathrm{e}^{-1} \quad \text{and} \quad B_2 = -1.$$

There is, therefore, only one solution, namely $y = \mathrm{e}^{x-1} - (1+x)$, that satisfies the two constraints $y(1) = -1$ and $y'(1) = 0$.

- **Initial value problem (IVP):** When all of the constraints are specified at the same value of x, the problem is called an initial value problem, as in *Example 2* above. In applications, initial value problems typically represent evolutionary problems in which the initial conditions specify the initial state of a system while the ODE describes its rate of change.

- **Boundary value problem (BVP):** When constraints are specified at two, or more, different values of x, for example at each end of an interval I, then the problem is called a boundary value problem, as in *Example 1* above. In applications, boundary value problems typically represent spatial problems in which the boundary conditions specify the state of a system at its boundary while the ODE describes its behaviour in the interior.

- **Note:** A first-order ordinary differential equation with one constraint is, automatically, an initial value problem.

- **Exceptions:** In these examples, we have seen that n independent constraints lead to a unique solution of an n^{th} order ordinary differential equation. However, this is not always the case!

1.3.4 Basic questions

- Given an initial value problem or a boundary value problem we would like to know the answers to the following questions:

  EXISTENCE: Is there any solution at all?
  An ODE arising from a physical problem should have at least one solution if the mathematical form of the model is correct.

  UNIQUENESS: How many solutions are there, or how many constraints are needed to obtain a unique solution?

  PROPERTIES: What are the properties of the solutions?
  Perhaps even without finding any solutions can we determine their general behaviour?
  How might different solutions be related to each other?

  SOLUTION: How can we find the solutions?
  (analytical methods, numerical techniques, power-series expansions, etc.)

- The final two questions are the main practical topic to be pursued in the rest of this course. The first two questions are partly answered by the existence and uniqueness theorem.

1.3.5 Existence and uniqueness

- **Theorem:** (***Existence and Uniqueness***)

  If $f(x, y)$ and $f_y(x, y)$ are continuous functions of x and y in a region $|x - \bar{x}| < a$ and $|y - \bar{y}| < b$, then there is only one solution $y = y(x)$, defined in some interval $|x - \bar{x}| < h \leq a$, which satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \quad \text{with} \quad y(\bar{x}) = \bar{y}.$$

  (f_y denotes the partial derivative of f with respect to y.)

- **Higher orders:** The theorem can be extended in a straightforward way to an n^{th} order ODE when n independent constraints (in the form of initial conditions) are required to guarantee existence and uniqueness.

- **Points to note:**
  - Only local existence and uniqueness are guaranteed for initial value problems.
  - The theorem says nothing about global existence or about existence and uniqueness for boundary value problems.
  - The existence and uniqueness theorem does not work in reverse. That is there are initial value problems with unique solutions for which the conditions of the theorem are violated.

1.3.6 Existence and uniqueness for linear ODEs

The existence and uniqueness theorem for linear ODEs has a much stronger form.

- **Theorem: (*Existence and Uniqueness for Linear ODEs*)** If $p(x)$ and $q(x)$ are continuous functions on an interval I, if $\bar{x} \in I$ and if $\bar{y} \in \mathbb{R}$, then there exists a unique solution $y = y(x)$ throughout the interval I for the ODE

  $$y' + p(x)\, y = q(x)$$

  which also satisfies the initial condition

  $$y(\bar{x}) = \bar{y}.$$

  This solution is a differentiable function and it satisfies the ODE throughout I.

- **Higher orders:** The theorem can be extended in a straightforward way to an n^{th} order linear ODE when n independent constraints (in the form of initial conditions) are required to guarantee existence and uniqueness.