Chapter 2

Analysis of strain

2.1 The infinitesimal strain tensor

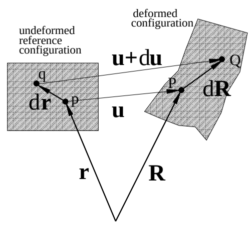

Figure 2.1: Sketch illustrating the deformation of an elastic body: The body is displaced, rotated and deformed.

- Lagrangian description: Label material points by their coordinates *before* the deformation (i.e. in the reference configuration).
- Displacement field: The material particle at position $r_i = x_i$ before the deformation is displaced to R_i after the deformation:
$$R_i = r_i + u_i(x_j). \tag{2.1}$$
- The deformation changes material line elements from dr_i $(= dx_i)$ to dR_i:
$$dR_i = dr_i + \left.\frac{\partial u_i}{\partial x_j}\right|_{\mathbf{r}} dx_j. \tag{2.2}$$
- We will restrict ourselves to a linearised anlysis in which the displacement derivatives are small, i.e.
$$\left|\frac{\partial u_i}{\partial x_j}\right| \ll 1. \tag{2.3}$$
- $\frac{\partial u_i}{\partial x_j}$ is the *displacement gradient tensor*:
$$\frac{\partial u_i}{\partial x_j} = e_{ij} + \omega_{ij}, \tag{2.4}$$
where
$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = e_{ji} \quad \text{is the } \textit{strain tensor} \text{ and} \tag{2.5}$$
$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = -\omega_{ji} \quad \text{is the } \textit{rotation tensor}. \tag{2.6}$$

- Displacements in the vicinity of $\mathbf{r}$:
$$u_i(\mathbf{r}+d\mathbf{r}) = \underbrace{u_i(\mathbf{r})}_{\substack{\text{Rigid Body}\\ \text{Translation}}} + \underbrace{\omega_{ij}dx_j}_{\substack{\text{Rigid Body}\\ \text{Rotation}}} + \underbrace{e_{ij}dx_j}_{\substack{\text{Pure}\\ \text{Deformation}}} \tag{2.7}$$

2.2 Rigid body rotation

- For $e_{ij} = 0$ (2.2) and (2.7):
$$d\mathbf{R} = d\mathbf{r} + \boldsymbol{\omega} \times d\mathbf{r} \tag{2.8}$$
where $\boldsymbol{\omega} = (\omega_{32}, \omega_{13}, \omega_{21})^T$. Represents rigid body rotation for $|\omega_{ij}| \ll 1$.

2.3 Pure deformation

2.3.1 Extensional deformation

- During the deformation the line element $dr_i = ds\ n_i$ is stretched to $dR_i = dS\ N_i$ ($\mathbf{n}$ and $\mathbf{N}$ are unit vectors).

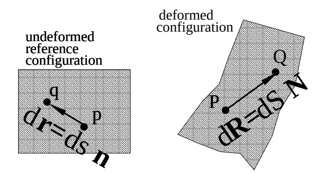

Figure 2.2: Sketch illustrating the extension (and rotation) of material line elements during the deformation of an elastic body.

- The *normal strain* $e_{\mathbf{n}}$ is the relative extension of the line element $ds\ \mathbf{n}$:
$$e_{\mathbf{n}} = \frac{dS - ds}{ds} = e_{ij} n_i n_j \tag{2.9}$$
- The $e_{(i)(i)}$ are the normal strains along the coordinate axes.

2.3.2 Shear deformation

- Consider the change of the angle between two material line elements $d\mathbf{r}^{(1)} = ds^{(1)}\mathbf{n}^{(1)}, d\mathbf{r}^{(2)} = ds^{(2)}\mathbf{n}^{(2)}$ which are orthogonal to each other in the undeformed state, $(dr_i^{(1)} dr_i^{(2)} = 0)$. Before the deformation: $\varphi = \pi/2$. After the deformation (see Fig. 2.3):
$$\cos\phi = 2 e_{ij} n_i^{(1)} n_j^{(2)}. \tag{2.10}$$
- The e_{ij} for $i \neq j$ are the *shear strains* w.r.t. the coordinate axes.

2.4 Principal axes/strain invariants

- The strain tensor gives the strains relative to the chosen coordinate system. Rotation of the coordinate system to a new one, such that
$$\tilde{x}_i = a_{ij} x_j \quad \text{where} \quad a_{ij} a_{kj} = \delta_{ik} \ \ (\text{orthogonal matrix, } \mathbf{A}^T = \mathbf{A}^{-1}) \tag{2.11}$$

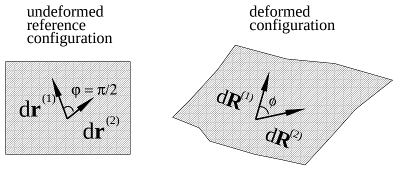

Figure 2.3: Sketch illustrating the shear deformation, i.e. the change in the angle between two material line elements during the deformation of an elastic body.

transforms the components of the strain tensor to:

$$\tilde{e_{ij}} = a_{ik} e_{kl} a_{jl} \quad \text{(symbolically } \tilde{\mathbf{E}} = \mathbf{A}\mathbf{E}\mathbf{A}^T\text{)}. \tag{2.12}$$

- There exists a special coordinate system (*principal axes*) in which $\tilde{e_{ij}} = 0$ for $i \neq j$.
- The principal axes are the normalised eigenvectors of e_{ij}.
- The normalised eigenvectors form the rows of the transformation matrix a_{ik} to the coordinate system formed by the principal axes.
- The eigenvalues of e_{ij} are the *principal strains*, i.e. the strains in the directions of the normal axes.
- The maximum normal strain, $\max e_{\mathbf{n}}$, (max. over all directions $\mathbf{n}$) is given by the maximum principal strain.
- The strain tensor has three invariants (i.e. quantities that are independent of the choice of the coordinate system):
  - **the dilation:** $d = e_{ii}$ which represents the relative change in volume
  $$d = e_{ii} = (dV - dv)/dv \tag{2.13}$$
  - **the determinant:** $\det e_{ij}$.
  - **and a third quantity:** $1/2(e_{ij}e_{ij} - e_{ii}e_{jj})$

2.5 Strain compatibility

- Equation (2.5) expresses e_{ij} in terms of a given displacement field u_i.
- The inverse problem: e_{ij} only describes a continuous deformation of a body (i.e. no gaps or overlaps of material develop during the deformation) iff:
$$e_{ij,kl} + e_{kl,ij} - e_{kj,il} - e_{il,kj} = 0 \tag{2.14}$$
This represents $3^4 = 81$ equations but only the ones corresponding to the following six parameter combinations are non-trivial and distinct:

i	1	1	1	1	1	2
j	1	1	2	1	2	2
k	2	2	2	3	3	3
l	2	3	3	3	3	3

- Geometrical interpretation which motivates the derivation of eqns. (2.14): e_{ij} determines the deformation of infinitesimal rectangular (cubic in 3D) blocks of material. After the deformation, the individually deformed blocks of material (deformed according to their local value of e_{ij}) must still fit together to form a continuous body.

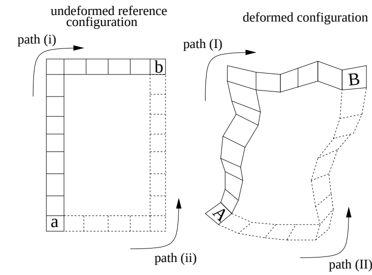

Figure 2.4: Sketch illustrating the strain compatibility condition.

2.6 Homogeneous deformation

- A deformation for which
$$\frac{\partial u_i}{\partial x_j} = const. \tag{2.15}$$
throughout the body is called a *homogeneous deformation*.

  Examples:

  Simple extension E.g. $e_{11} = e_0$, $e_{ij} = 0$ otherwise.

  Uniform dilation $e_{ij} = e_0 \delta_{ij}$ (spherically symmetric).

  Simple shearing E.g. $e_{12} = e_{21} = e_0$, $e_{ij} = 0$ otherwise.