Chapter 3

Analysis of stress

3.1 The concept of traction/stress

- If $\Delta\mathcal{F}$ is the resultant force acting on a small area element ΔS with unit normal $\mathbf{n}$, then the traction (stress) vector $\mathbf{t}$ is defined as:

$$\mathbf{t} = \lim_{\Delta S \to 0} \frac{\Delta \boldsymbol{\mathcal{F}}}{\Delta S} \tag{3.1}$$

The term 'traction' is usually used for stresses acting on the surfaces of a body.

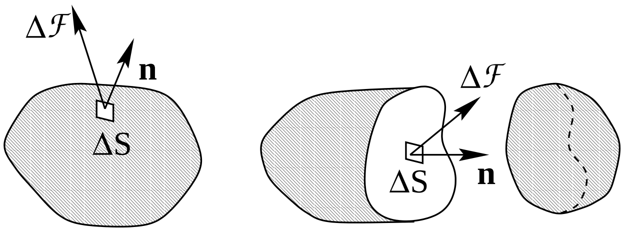

Figure 3.1: Sketch illustrating traction and stress.

3.2 The stress tensor

- The stress vector $\mathbf{t}$ depends on the spatial position in the body and on the orientation of the plane (characterised by the normal vector):

$$t_i = \tau_{ij} n_j, \tag{3.2}$$

where $\tau_{ij} = \tau_{ji}$ is the *stress tensor.*

- On an infinitesimal block of material whose faces are parallel to the axes, the component τ_{ij} of the stress tensor represents the traction component in the positive i-direction on the face $x_j = const.$ whose normal points in the positive j-direction (see Fig. 3.2).

3.3 The equations of equilibrium/motion

- The equations of equilibrium for a body, subject to a body force (force per unit volume) F_i is

$$\frac{\partial \tau_{ij}}{\partial x_j} + F_i = 0. \tag{3.3}$$

- Including inertial effects via D'Alembert forces gives the equations of motion:

$$\frac{\partial \tau_{ij}}{\partial x_j} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \tag{3.4}$$

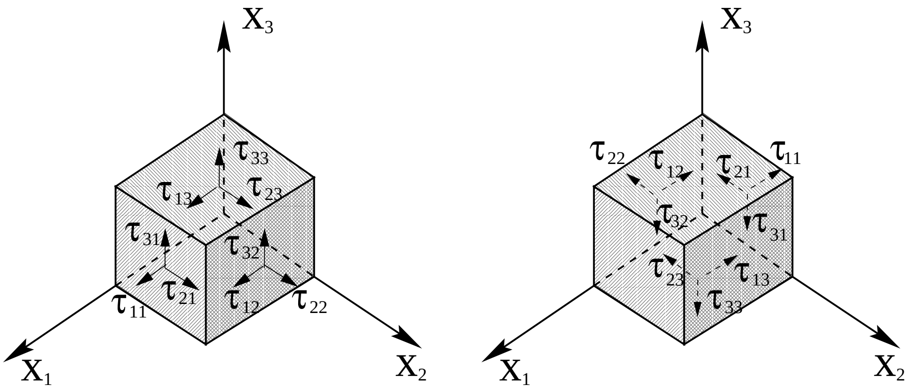

Figure 3.2: Sketch illustrating the components of the stress tensor.

where ρ is the density of the body and t is time.

3.4 Principal axes/stress invariants

- The stress tensor is real and symmetric, hence all considerations in section 2.4 apply to the stress tensor as well (transformation to different coordinate systems, principal axes, max. stress and invariants).
- In particular, we will denote the first invariant (the trace of the stress vector) by

$$\theta = \tau_{ii}. \tag{3.5}$$

3.5 Homogeneous stress states

- Analogous to homogeneous deformations (see section 2.6): Examples:

  Uniaxial stress E.g. $\tau_{11} = T_0$, $\tau_{ij} = 0$ otherwise.

  Hydrostatic pressure $\tau_{ij} = P_0\delta_{ij}$ (spherically symmetric).

  Pure shear stress E.g. $\tau_{12} = \tau_{21} = T_0$, $\tau_{ij} = 0$ otherwise.