Chapter 6

Plane strain problems

6.1 Basic equations

Definition: A deformation is said to be one of *plane strain* (parallel to the plane $x_3 = 0$) if:

$$u_3 = 0 \quad \text{and} \quad u_\alpha = u_\alpha(x_\beta). \tag{6.1}$$

- There are only two independent variables, $(x_1, x_2) = (x, y)$.
- Plane strain is only possible if $F_3 = 0$.
- Only the in-plane strains are non-zero, $e_{i3} = 0$.
- Stress-strain relationship:

$$\tau_{\alpha\beta} = \lambda \delta_{\alpha\beta} e_{\gamma\gamma} + 2\mu e_{\alpha\beta}. \tag{6.2}$$

$$2\mu e_{\alpha\beta} = \tau_{\alpha\beta} - \nu \delta_{\alpha\beta} \underbrace{\tau_{\gamma\gamma}}_{\tilde{\theta}} \tag{6.3}$$

$$\tau_{33} = \nu \tau_{\gamma\gamma} = \nu \tilde{\theta} \tag{6.4}$$

- Static equilibrium equations:

$$\tau_{\alpha\beta,\beta} + F_\alpha = 0 \tag{6.5}$$

- Compatibility equation: Only one non-trivial equation

$$0 = e_{11,22} + e_{22,11} - 2e_{12,12} \tag{6.6}$$

  Formulated in terms of stresses:

$$(1-\nu)\tilde{\theta}_{,\alpha\alpha} + F_{\alpha,\alpha} = 0, \tag{6.7}$$

  or symbolically

$$(1-\nu)\tilde{\nabla}^2 \tilde{\theta} + \operatorname{div} \mathbf{F} = 0, \tag{6.8}$$

  where $\tilde{\nabla}^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

6.2 The Airy stress function

- For $\mathbf{F} = 0$ the in-plane stresses can be expressed in terms of the *Airy stress function* ϕ:

$$\tau_{11} = \frac{\partial^2 \phi}{\partial y^2}, \quad \tau_{22} = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{12} = -\frac{\partial^2 \phi}{\partial x \partial y}. \tag{6.9}$$

- The Airy stress function is *biharmonic*:

$$\tilde{\nabla}^4 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0. \tag{6.10}$$

6.3 The stress boundary conditions in terms of the Airy stress function

- The applied tractions along the boundary ∂D (parametrised by the arclength s) are given in terms of the Airy stress function ϕ by
$$t_1(s) = t_x(s) = \frac{d}{ds}\left(\frac{\partial \phi}{\partial y}\right) \tag{6.11}$$
and
$$t_2(s) = t_y(s) = -\frac{d}{ds}\left(\frac{\partial \phi}{\partial x}\right). \tag{6.12}$$

- Hence, if $t_\alpha(s)$ is given, the boundary conditions for ϕ can be derived by the following procedure:
  1. Integrate (6.11) and (6.12) along the boundary (w.r.t. s). This provides $(\partial\phi/\partial x, \partial\phi/\partial y)^T = \tilde{\nabla}\phi$ on the boundary.
  2. Rewrite $\tilde{\nabla}\phi = \partial\phi/\partial s \ \mathbf{e}_t + \partial\phi/\partial n \ \mathbf{e}_n$ where $\mathbf{e}_t$ and $\mathbf{e}_n$ are the unit tangent and (outer) normal vectors on the boundary. This provides $\partial\phi/\partial s$ and $\partial\phi/\partial n$ along the boundary.
  3. Integrate $\partial\phi/\partial s$ along the boundary (w.r.t. s). This provides ϕ along the boundary.

- After this procedure ϕ and $\partial\phi/\partial n$ are known along the entire boundary and can be used as the boundary condition for the fourth order biharmonic equation (6.10).

- Note: any constants of integration arising during the procedure can be set to zero.

- For a traction free boundary, $t_\alpha(s) = 0$, we can use the boundary conditions:
$$\phi = 0 \quad \text{and} \quad \partial\phi/\partial n = 0 \quad \text{on } \partial D \tag{6.13}$$

6.4 The displacements in terms of the Airy stress function

- For a given Airy stress function $\phi(x, y)$, the displacements $u(x, y), v(x, y)$, are determined by
$$2\mu\frac{\partial u}{\partial x} = (1-\nu)\tilde{\nabla}^2\phi - \frac{\partial^2\phi}{\partial x^2}, \tag{6.14}$$
$$2\mu\frac{\partial v}{\partial y} = (1-\nu)\tilde{\nabla}^2\phi - \frac{\partial^2\phi}{\partial y^2} \tag{6.15}$$
and
$$\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -\frac{\partial^2\phi}{\partial x \partial y}. \tag{6.16}$$

- One way to determine the displacement fields from these equations is given by the following procedure:
  1. Get $p(x, y) = \tilde{\nabla}^2\phi(x, y)$ from the known $\phi(x, y)$.
  2. $p(x, y)$ is a harmonic function; determine its complex conjugate $q(x, y)$ from the Cauchy-Riemann equations:
  $$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} \quad \text{and} \quad \frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x}. \tag{6.17}$$
  3. Integrate $f(z) = f(x + iy) = p(x, y) + i\ p(x, y)$ and thus determine $P(x, y)$ and $Q(x, y)$ from
  $$F(z) = \int f(z)dz =: P(x, y) + i\ Q(x, y). \tag{6.18}$$

4. Then the displacements are given by:

$$u(x,y) = \frac{1}{2\mu}\Big[(1-\nu)\ P(x,y) - \frac{\partial\phi}{\partial x} + \underbrace{a+cy}_{\text{rigid body motion}}\Big] \tag{6.19}$$

and

$$v(x,y) = \frac{1}{2\mu}\Big[(1-\nu)\ Q(x,y) - \frac{\partial\phi}{\partial y} + \underbrace{b-cx}_{\text{rigid body motion}}\Big]. \tag{6.20}$$

6.5 Equations in polar coordinates

- The biharmonic equation in polar coordinates:

$$\tilde{\nabla}^4\phi(r,\varphi) = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right]\left[\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2}\right] \tag{6.21}$$

$$\tilde{\nabla}^4\phi(r,\varphi) = \phi_{,rrrr} + \frac{2}{r}\phi_{,rrr} - \frac{1}{r^2}(\phi_{,rr} - 2\phi_{,rr\varphi\varphi}) + \frac{1}{r^3}(\phi_{,r} - 2\phi_{,r\varphi\varphi}) + \frac{1}{r^4}(4\phi_{,\varphi\varphi} + 2\phi_{,\varphi\varphi\varphi\varphi}) \tag{6.22}$$

- For axisymmetric solutions:

$$\tilde{\nabla}^4\phi(r) = \frac{1}{r}\left[r\left(\frac{1}{r}[r\phi_{,r}]_{,r}\right)_{,r}\right]_{,r} \tag{6.23}$$

$$\tilde{\nabla}^4\phi(r) = \phi_{,rrrr} + \frac{2}{r}\phi_{,rrr} - \frac{1}{r^2}\phi_{,rr} + \frac{1}{r^3}\phi_{,r} \tag{6.24}$$

- Stresses:

$$\tau_{rr} = \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}, \tag{6.25}$$

$$\tau_{\varphi\varphi} = \frac{\partial^2\phi}{\partial r^2} \tag{6.26}$$

$$\tau_{r\varphi} = \frac{1}{r^2}\frac{\partial\phi}{\partial\varphi} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\varphi} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\varphi}\right). \tag{6.27}$$

6.6 Particular solutions of the biharmonic equation

6.6.1 Harmonic functions

- Obviously, all harmonic functions also fulfil the biharmonic equation.

6.6.2 Power series expansions

$$\phi(x,y) = \sum_{i,k} a_{ik}x^i y^k \tag{6.28}$$

- Any terms with $i+k<2$ do not give a contribution.
- Any terms with $i+k<4$ fulfil $\tilde{\nabla}^4\phi = 0$ for arbitrary constants a_{ik}. Special cases are:

$\phi(x,y)$	τ_{xx}	τ_{yy}	τ_{xy}	Interpretation:
$a_{02}\ y^2$	$2\ a_{02}$	0	0	constant tension in x-direction
$a_{11}\ xy$	0	0	$-a_{11}$	pure shear
$a_{20}\ x^2$	0	$2\ a_{20}$	0	constant tension in y-direction
$a_{03}\ y^3$	$6\ a_{03}y$	0	0	pure x-bending
$a_{30}\ x^3$	0	$6\ a_{30}x$	0	pure y-bending

- Linear combinations provide stress fields for combined load cases.

6.6.3 Solutions in polar coordinates

- The general axisymmetric solution:

$$\phi(r) = A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r \tag{6.29}$$

- The general separated non-axisymmetric solution:

For $n = 1$**:**

$$\begin{aligned}\phi(r,\varphi) = &\left(Ar + \frac{B}{r} + Cr^3 + Dr\log r\right) \quad \cos(\varphi)\\ &+ \left(ar + \frac{b}{r} + cr^3 + dr\log r\right) \quad \sin(\varphi)\end{aligned} \tag{6.30}$$

For $n \geq 2$**:**

$$\begin{aligned}\phi(r,\varphi) = \sum_{n=2}^{\infty} &\left(A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2}\right) \quad \cos(n\varphi)\\ &+ \left(a_n r^n + b_n r^{-n} + c_n r^{n+2} + d_n r^{-n+2}\right) \quad \sin(n\varphi)\end{aligned} \tag{6.31}$$

6.7 St. Venant's principle

Section 6.6 provides many solutions of the biharmonic equation. The free constants in these solutions have to be determined from the boundary conditions. This is the hardest part of the solution! 'Good' approximate solutions can often be obtained by using:

> *St. Venant's principle*
>
> In elastostatics, if the boundary tractions $\mathbf{t}$ on a part ∂D_1 of the boundary ∂D are replaced by a statically equivalent traction distribution $\hat{\mathbf{t}}$, the effects on the stress distribution in the body are negligible at points whose distance from ∂D_1 is large compared to the maximum distance between the points of ∂D_1.

'Statically equivalent' means that the resultant forces and moments due to the two tractions $\mathbf{t}$ and $\hat{\mathbf{t}}$ are identical. Hence, the traction boundary conditions are not fulfilled pointwise but in an average sense.