Chapter 5

The equations of linear elasticity

5.1 Summary of equations

- Strain-displacement relations:
$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{5.1}$$
- Equilibrium equations/equations of motion:
$$\tau_{ij,j} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{5.2}$$
- Constitutive equations:
$$\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} \tag{5.3}$$

5.2 Displacement formulation: The Navier-Lamé equations

- Solve for the displacements:
$$(\lambda + \mu) u_{k,ki} + \mu u_{i,kk} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{5.4}$$
or symbolically:
$$(\lambda + \mu) \text{ grad div } \mathbf{u} + \mu \nabla^2 \mathbf{u} + \mathbf{F} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \tag{5.5}$$
which is equivalent to:
$$(\lambda + 2\mu) \text{ grad div } \mathbf{u} - \mu \text{ curl curl } \mathbf{u} + \mathbf{F} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{5.6}$$
- This is a system of three coupled linear elliptic PDEs for the three displacements $u_i(x_j)$.

5.3 Stress formulation: The static Beltrami-Mitchell equations

- For static deformations, we have
$$\frac{1-\nu}{1+\nu} \underbrace{\tau_{ii,jj}}_{\theta_{,jj}} + F_{i,i} = 0 \quad \text{or symbolically} \quad \frac{1-\nu}{1+\nu} \nabla^2 \theta + \text{div } \mathbf{F} = 0. \tag{5.7}$$
and the stresses fulfil the Beltrami-Mitchell equations:
$$\underbrace{\tau_{ij,kk}}_{\nabla^2 \tau_{ij}} + \frac{1}{1+\nu} \underbrace{\tau_{kk,ij}}_{\theta_{,ij}} + \frac{\nu}{1-\nu} \delta_{ij} \underbrace{F_{k,k}}_{\text{div } \mathbf{F}} + F_{j,i} + F_{i,j} = 0. \tag{5.8}$$
- (5.8) represents a system of six coupled linear elliptic PDEs for the six stress components $\tau_{ij}(x_j)$. When these have been determined, the strains can be recovered from (4.6) or (4.16). Then the displacements follow from (5.1). They are only determined up to arbitrary rigid body motions.

5.4 Simplifications for $\mathbf{F} = const.$:

- For constant (or vanishing!) body force, the stress, strain and displacement components are biharmonic functions,
$$u_{i,jjkk} = 0 \qquad \tau_{ij,kkll} = 0 \qquad e_{ij,kkll} = 0 \tag{5.9}$$
or symbolically:
$$\nabla^4 \mathbf{u} = 0 \qquad \nabla^4 \tau_{ij} = 0 \qquad \nabla^4 e_{ij} = 0. \tag{5.10}$$

- The dilation and the trace of the stress tensor are harmonic functions:
$$u_{j,jkk} = d_{,kk} = 0 \qquad \tau_{jj,kk} = \theta_{,kk} = 0 \tag{5.11}$$
or symbolically:
$$\nabla^2 d = 0 \qquad \nabla^2 \theta = 0 \tag{5.12}$$

- Note that in (5.4) – (5.8) $\mathbf{F}$ acts as an inhomogeneity in a system of linear equations. The system can be transformed into a homogeneous system for $\mathbf{u}_h = \mathbf{u} - \mathbf{u}_p$ (with different boundary conditions) if a particular solution $\mathbf{u}_p$ (which does not have to fulfil the boundary conditions) can be found.

5.5 Boundary conditions:

- Displacement (Dirichlet) boundary conditions: Prescribed displacement field $u_i^{(0)}$.
$$u_i|_{\partial D} = u_i^{(0)} \tag{5.13}$$

- Stress (Neumann) boundary conditions: Prescribed (applied) traction $t_i^{(0)}$ on boundary. Note that n_j is the *outer* unit normal vector on the elastic body.
$$t_i|_{\partial D} = \tau_{ij} n_j|_{\partial D} = t_i^{(0)} \tag{5.14}$$

- Mixed (Robin) boundary conditions – 'elastic foundation' represented by the stiffness tensor k_{ij}. Physically, this implies that the traction which the elastic foundation exerts on the body is proportional to the boundary displacement. This can be combined with an applied traction $t_i^{(0)}$ as in the Neumann case.
$$(t_i + k_{ij} u_j)\,|_{\partial D} = (\tau_{ij} n_j + k_{ij} u_j)\,|_{\partial D} = t_i^{(0)} \tag{5.15}$$

Governing Equations in Cylindrical Polar Coordinates

- $x_1 = x = r\cos\theta$, $x_2 = y = r\sin\theta$, $x_3 = z = z$.

$$\mathbf{u} = (u_r, u_\theta, u_z), \quad e = (e_{ij}), \quad \tau = (\tau_{ij}), \quad \text{where } i, j = r, \theta, z.$$

- Vector calculus:

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, \qquad \operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

$$\operatorname{curl}\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial(ru_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{z}}.$$

- Stress-strain relations have the same form as in Cartesian coordinates:

$$\tau_{ij} = \lambda\delta_{ij}\operatorname{div}\mathbf{u} + 2\mu e_{ij}, \qquad i, j = r, \theta, z.$$

- Stress-displacement relations:

$$\tau_{rr} = \lambda\operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda\operatorname{div}\mathbf{u} + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right), \quad \tau_{zz} = \lambda\operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_z}{\partial z},$$

$$\frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad \frac{\tau_{rz}}{\mu} = \frac{\tau_{zr}}{\mu} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \quad \frac{\tau_{\theta z}}{\mu} = \frac{\tau_{z\theta}}{\mu} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

- Strain-displacement relations:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \qquad e_{zz} = \frac{\partial u_z}{\partial z},$$

$$2e_{r\theta} = 2e_{\theta r} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad 2e_{rz} = 2e_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \quad 2e_{z\theta} = 2e_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

- Equilibrium equations (statics): for the displacement formulation, use Navier's equation,

$$(\lambda + 2\mu)\operatorname{grad}\operatorname{div}\mathbf{u} - \mu\operatorname{curl}\operatorname{curl}\mathbf{u} + \mathbf{F} = \mathbf{0},$$

whereas for the stress formulation, use

$$\begin{aligned}
\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} + F_r &= 0 \\
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} + F_\theta &= 0 \\
\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r}\tau_{rz} + F_z &= 0.
\end{aligned}$$

- Stress boundary conditions: these are when $\mathbf{T}$ is prescribed. We have, from $T_i = \hat{n}_j\tau_{ij}$,

$$\begin{aligned}
T_r &= \hat{n}_r\tau_{rr} + \hat{n}_\theta\tau_{r\theta} + \hat{n}_z\tau_{rz} \\
T_\theta &= \hat{n}_r\tau_{r\theta} + \hat{n}_\theta\tau_{\theta\theta} + \hat{n}_z\tau_{\theta z} \\
T_z &= \hat{n}_r\tau_{rz} + \hat{n}_\theta\tau_{\theta z} + \hat{n}_z\tau_{zz}
\end{aligned}$$

Governing Equations in Spherical Polar Coordinates

- $x_1 = x = r\sin\theta\,\cos\phi$, $x_2 = y = r\sin\theta\,\sin\phi$, $x_3 = z = r\cos\theta$.

$$\mathbf{u} = (u_r, u_\theta, u_\phi), \quad e = (e_{ij}), \quad \tau = (\tau_{ij}), \quad \text{where } i,j = r,\theta,\phi.$$

- Vector calculus:

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}},$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta\,u_r) + \frac{\partial}{\partial\theta}(r\sin\theta\,u_\theta) + \frac{\partial}{\partial\phi}(ru_\phi)\right\},$$

$$\operatorname{curl}\mathbf{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ u_r & ru_\theta & r\sin\phi\,u_\phi \end{vmatrix}.$$

- Stress-strain relations have the same form as in Cartesian coordinates:

$$\tau_{ij} = \lambda\delta_{ij}\operatorname{div}\mathbf{u} + 2\mu e_{ij}, \qquad i,j = r,\theta,\phi.$$

- Stress-displacement relations:

$$\tau_{rr} = \lambda\operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda\operatorname{div}\mathbf{u} + \frac{2\mu}{r}\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right),$$

$$\tau_{\phi\phi} = \lambda\operatorname{div}\mathbf{u} + \frac{2\mu}{r}\left(\frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial\phi} + u_r + u_\theta\cot\theta\right), \quad \frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta},$$

$$\frac{\tau_{r\phi}}{\mu} = \frac{\tau_{\phi r}}{\mu} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}, \quad \frac{\tau_{\theta\phi}}{\mu} = \frac{\tau_{\phi\theta}}{\mu} = \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial\theta} - \frac{u_\phi\cot\theta}{r}.$$

- Strain-displacement relations:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \qquad e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r},$$

$$2e_{r\theta} = 2e_{\theta r} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}, \quad 2e_{r\phi} = 2e_{\phi r} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r},$$

$$2e_{\phi\theta} = 2e_{\theta\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial\theta} - \frac{u_\phi\cot\theta}{r}.$$

- Equilibrium equations (statics): for the displacement formulation, use Navier's equation,

$$(\lambda+2\mu)\operatorname{grad}\operatorname{div}\mathbf{u} - \mu\operatorname{curl}\operatorname{curl}\mathbf{u} + \mathbf{F} = \mathbf{0},$$

whereas for the stress formulation, use

$$\begin{aligned}
\frac{\partial\tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{r\phi}}{\partial\phi} + \frac{2\tau_{rr} - \tau_{\theta\theta} - \tau_{\phi\phi} + \cot\theta\,\tau_{r\theta}}{r} + F_r &= 0 \\
\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{3\tau_{r\theta} + (\tau_{\theta\theta} - \tau_{\phi\phi})\cot\theta}{r} + F_\theta &= 0 \\
\frac{\partial\tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi\phi}}{\partial\phi} + \frac{3\tau_{r\phi} + 2\tau_{\theta\phi}\cot\theta}{r} + F_\phi &= 0.
\end{aligned}$$

- Stress boundary conditions: these are when $\mathbf{T}$ is prescribed. We have, from $T_i = \hat{n}_j\tau_{ij}$,

$$\begin{aligned}
T_r &= \hat{n}_r\tau_{rr} + \hat{n}_\theta\tau_{r\theta} + \hat{n}_\phi\tau_{r\phi} \\
T_\theta &= \hat{n}_r\tau_{r\theta} + \hat{n}_\theta\tau_{\theta\theta} + \hat{n}_\phi\tau_{\theta\phi} \\
T_\phi &= \hat{n}_r\tau_{r\phi} + \hat{n}_\theta\tau_{\theta\phi} + \hat{n}_\phi\tau_{\phi\phi}
\end{aligned}$$