Chapter 4

Elasticity & constitutive equations

4.1 The constitutive equations

- The constitutive equations determine the stress τ_{ij} in the body as function of the body's deformation.

Definition: A solid body is called *elastic* if

$$\tau_{ij}(x_n, t) = \tau_{ij}(e_{kl}(x_n, t)). \tag{4.1}$$

i.e. if the stress depends on the *instantaneous, local* values of the strain only.

- For *small* strains, a Taylor expansion of (4.1) gives:

$$\tau_{ij} = \underbrace{\tau_{ij}\big|_{e_{kl}=0}}_{\text{Initial Stress } \tau_{ij}^0} + \underbrace{\frac{\partial \tau_{ij}}{\partial e_{kl}}\bigg|_{e_{kl}=0}}_{E_{ijkl}} e_{kl}. \tag{4.2}$$

- If the reference configuration coincides with a stress free state, then $\tau_{ij}^0 = 0$ and we obtain *Hooke's law*:

$$\tau_{ij} = E_{ijkl} e_{kl}. \tag{4.3}$$

Definition: A solid body is called *homogenous* if E_{ijkl} is independent of x_i.

Definition: A solid body is called *isotropic* if its elastic properties are the same in all directions.

- For an isotropic homogeneous elastic solid:

$$E_{ijkl} = \lambda \delta_{ij} \delta_{kl} + 2\mu \delta_{ik} \delta_{jl}, \tag{4.4}$$

where λ and μ are the *Lamé constants*.

- Stress-strain relationship for an isotropic homogeneous elastic solid:

$$\tau_{ij} = \lambda \delta_{ij} \underbrace{e_{kk}}_{=d} + 2\mu e_{ij}, \tag{4.5}$$

and in the inverse form:

$$e_{ij} = \underbrace{\frac{1}{2\mu}\left(\delta_{ik}\delta_{jl} - \frac{\lambda}{(3\lambda + 2\mu)}\delta_{ij}\delta_{kl}\right)}_{D_{ijkl}} \tau_{kl} \tag{4.6}$$

so that

$$e_{ij} = D_{ijkl} \tau_{kl}. \tag{4.7}$$

Written out:

$$e_{ij} = \frac{1}{2\mu}\tau_{ij} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\delta_{ij} \underbrace{\tau_{kk}}_{=\theta} \tag{4.8}$$

- For an isotropic homogeneous elastic solid the principal axes of the stress and strain tensors coincide and

$$\theta = \tau_{kk} = (3\lambda + 2\mu)d = (3\lambda + 2\mu)e_{kk} \tag{4.9}$$

4.2 Experimental determination of elastic constants

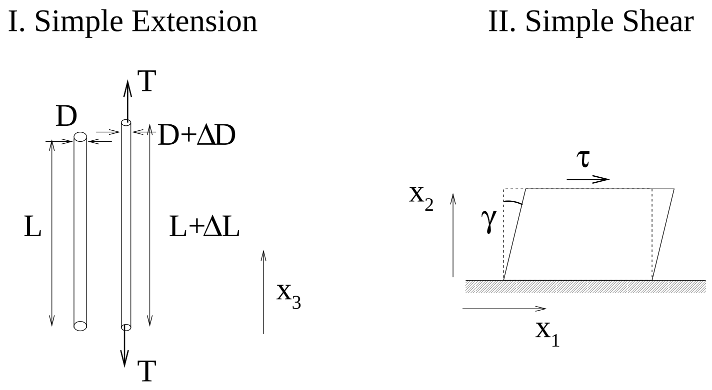

Figure 4.1: Sketch illustrating the two fundamental experiments for the determination of the elastic constants.

4.2.1 Experiment I: Simple extension of a thin cylinder

- Observations:

$$T = EA\frac{\Delta L}{L} \tag{4.10}$$

i.e.

$$\tau_{33} = Ee_{33} \tag{4.11}$$

(since $e_{33} = \Delta L/L$) and

$$\frac{e_{11}}{e_{33}} = \frac{e_{22}}{e_{33}} = -\nu \tag{4.12}$$

where $e_{11} = e_{22} = \Delta D/D$.

- E and ν are *Young's modulus* and *Poisson's ration*, respectively.

4.2.2 Experiment II: Simple shear

- Observation:

$$\tau = G\gamma \tag{4.13}$$

i.e.

$$\tau_{12} = G\ 2e_{12}. \tag{4.14}$$

- G is the material's *shear modulus*.

4.2.3 Constitutive equations in terms of E and ν

$$\tau_{ij} = \frac{E}{1+\nu}\left(e_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\underbrace{e_{kk}}_{d}\right). \tag{4.15}$$

- Note that materials with $\nu = 1/2$ are *incompressible*, i.e. $d \equiv 0$.

$$e_{ij} = \frac{1}{E}\left((1+\nu)\tau_{ij} - \nu\delta_{ij}\underbrace{\tau_{kk}}_{\theta}\right). \tag{4.16}$$

4.3 Relations between the elastic constants

	$\lambda =$	$\mu = G =$	$E =$	$\nu =$
λ, μ	λ	μ	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$
λ, ν	λ	$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{(1+\nu)(1-2\nu)\lambda}{\nu}$	ν
μ, E	$\frac{\mu(E-2\mu)}{3\mu-E}$	μ	E	$\frac{E-2\mu}{2\mu}$
E, ν	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	E	ν