MT4271: SOLUTION FOR EXAMPLE SHEET[1] I (for Oct. 1, 1998)

1.) Which one of these equations in index notation are valid? Remember the summation convention!

a) $c = a_i b_i$ (OK, this is the dot product $c = \mathbf{a} \cdot \mathbf{b}$)

b) $c = a_{ij} b_i$ (Wrong, the free index j doesn't appear on LHS)

c) $c_i = a_{ij} b_i$ (Wrong, the indices on LHS and RHS don't match)

d) $c_i = a_{ij} b_j$ (OK, this is the matrix vector product with the matrix $\underline{\mathbf{a}}$: $\mathbf{c} = \underline{\mathbf{a}}\mathbf{b}$)

e) $c_i = a_{ji} b_j$ (OK, this is the matrix vector product with the transposed matrix $\underline{\mathbf{a}}$: $\mathbf{c} = \underline{\mathbf{a}}^T\mathbf{b}$)

f) $\sigma_{ij} = \alpha_{ij} T + E_{ijkl} e_{kl}$ (Correct – meet your first 4th order tensor. By the way: this is the constitutive equation for a linearly elastic solid incl. temperature variations)

g) $\sigma_{ij} = \alpha_{kl} T_i + E_{ijkl} e_{ij}$ (Wrong, the indices of all terms are different)

h) $k_{ijkl} = a_i b_{kl} c_{njm} d_{mn} + e_{ik} e_{jn} f_{nl}$ (Messy, but correct)

2.) Using a comma to denote partial differentiation (e.g. $\partial u / \partial x_2 = u_{,2}$), transform the following expressions into index notation:

a) $\nabla u(x_1, x_2, x_3) \rightarrow u_{,i}$

b) $\underline{\mathbf{A}} = \nabla \mathbf{u}(x_1, x_2, x_3) \rightarrow a_{ij} = u_{i,j}$

c) $\nabla \cdot \mathbf{u}(x_1, x_2, x_3) = f(x_1, x_2, x_3) \rightarrow u_{i,i} = f$

d) $\nabla^2 u(x_1, x_2, x_3) = f(x_1, x_2, x_3) \rightarrow u_{,ii} = f$

e) $\nabla^2 \mathbf{u}(x_1, x_2, x_3) = \mathbf{f}(x_1, x_2, x_3) \rightarrow u_{i,jj} = f_i$

3.) A 2D body occupying the region $\{d : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$ is displaced by the following displacement fields:

a) $u_1 = \epsilon(x_1 + 2x_2)$; $u_2 = \epsilon(3 + x_2)$ where $\epsilon \ll 1$

b) $u_1 = U_1 - x_1 + \sqrt{x_1^2 + x_2^2}\, \cos\left(\arctan\left(\frac{x_2}{x_1}\right) + \Phi\right)$;
$u_2 = U_2 - x_2 + \sqrt{x_1^2 + x_2^2} \sin\left(\arctan\left(\frac{x_2}{x_1}\right) + \Phi\right)$ where U_1, U_2, Φ are constants.

Sketch the deformed body D (setting $\epsilon = 1$ in (a) for simplicity). For the displacement field (a) determine the deformation gradient tensor and the strain and rotation tensors. For the displacement field (b) determine the linear strain and rotation tensors for $\Phi \ll 1$ – Think before you calculate!

Solution:

[1] Any feedback to: *M.Heil@maths.man.ac.uk*

a) For $\epsilon = 1$:

$$\mathbf{R} = \mathbf{r} + \mathbf{u} = (2(x_1 + x_2),\ 3 + 2x_2)^T$$

Use this to sketch the edges.

For arbitrary (small) ϵ:

$$u_{i,j} = \epsilon \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},\ e_{ij} = \epsilon \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ \omega_{ij} = \epsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

b) The body is displaced by $(U_1, U_2)^T$ and then rotated by Φ (about the $\mathbf{e}_3$ axis). Therefore for small rotations and a linearised theory:

$$e_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ \omega_{ij} = \begin{pmatrix} 0 & -\Phi \\ \Phi & 0 \end{pmatrix}$$

This can be confirmed by linearisation of e_{ij} and ω_{ij} w.r.t. Φ. (Don't do it by hand, though!)