# Chapter 1

# Introduction

This set of notes summarises the main results of the lecture 'Elasticity' (MT4271). Please email any corrections (yes, there might be the odd typo...) or suggestions for improvement to *M.Heil@maths.man.ac.uk* or see me after the lecture or in my office (18.07).

Generally, the notes will be handed out after the material has been covered in the lecture. You can also download them from the WWW:

*http://www.maths.man.ac.uk/ ˜mheil/Lectures/Elasticity/MT4271-Elasticity.html.*

This WWW page will also contain announcements, example sheets, solutions, etc.

## 1.1 Literature

The following is a list of books that I found useful in preparing this lecture. I've quoted the prices where I knew what they were. **It is not necessary to purchase any of these books!** Your lecture notes and these handouts will be completely sufficient.

**Textbook which covers most of the material in this lecture:** Gould, P.L. *Introduction to Linear Elasticity, 2nd ed.* Springer (1994) – £51

**Nice (useful) review of Linear Algebra:** Banchoff, T. & Wermer, J. *Linear Algebra Through Geometry, 2nd ed.* Springer (1991).

**One of the classic elasticity texts:** Green, A.E. & Zerna, W. *Theoretical Elasticity.* Dover (1992) – paperback reprint of the original version from Oxford University press £11.95

**And another classic:** Love, A.E.H. *Treatise on the Mathematical Theory of Elasticity.* Dover (1944) – paperback reprint of the original version from Cambridge University press £15.95

**A beautiful little book (but out of print!):** Long, R.R. *Mechanics of Solids and Fluids.* Prentice-Hall, (1961) – £11.00 (back then, presumably...). Try the library.

**My own favourite elasticity book (this book saved my PhD!):** Wempner, G. *Mechanics of Solids with Applications to Thin Bodies.* Kluwer Academic Publishers Group (1982) – unfortunately only available as hardback for £126!!

## 1.2 Preliminaries: Index notation & summation convention

- Denote vectors/matrices/tensors by their components, i.e. $\mathbf{r} = r_i$; $\mathbf{A} = A_{ij}$
- Greek indices range from 1 to 2; Latin ones from 1 to 3.
- Kronecker Delta: $\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$
- Summation convention: Automatic summation over repated indices. E.g.:





**Dot product:** $\mathbf{a} \cdot \mathbf{b} = a_i b_i = a_k b_k$ (Dummy index!)

$\delta_{ij}$ **'exchanges' indices:** $a_i \delta_{ij} = a_j$.

**Matrix-vector products:** $\mathbf{A} \cdot \mathbf{x} = A_{ij} x_j = A_{im} x_m$ $\qquad \mathbf{A}^T \cdot \mathbf{x} = A_{ji} x_j$

**No summation over indices in brackets:** E.g. diagonal matrix: $diag(\lambda_1, \lambda_2, \lambda_3) = \lambda_{(i)} \delta_{(i)j}$.

- Comma denotes partial differentiation: E.g. $\frac{\partial u_i}{\partial x_j} = u_{i,j}$.
- Some differential operators in index notation:

$$\nabla \cdot \mathbf{u} = \text{div } \mathbf{u} = u_{i,i} \tag{1.1}$$

$$\nabla \phi = \text{grad}\phi = \phi_{,i} \tag{1.2}$$

$$\nabla^2 \phi = \phi_{,ii} \tag{1.3}$$

# Chapter 2

# Analysis of strain

## 2.1 The infinitesimal strain tensor

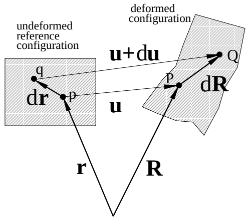


Figure 2.1: Sketch illustrating the deformation of an elastic body: The body is displaced, rotated and deformed.

- Lagrangian description: Label material points by their coordinates *before* the deformation (i.e. in the reference configuration).
- Displacement field: The material particle at position $r_i = x_i$ before the deformation is displaced to $R_i$ after the deformation:
$$R_i = r_i + u_i(x_j). \tag{2.1}$$
- The deformation changes material line elements from $dr_i$ $(= dx_i)$ to $dR_i$:
$$dR_i = dr_i + \left.\frac{\partial u_i}{\partial x_j}\right|_{\mathbf{r}} dx_j. \tag{2.2}$$
- We will restrict ourselves to a linearised anlysis in which the displacement derivatives are small, i.e.
$$\left|\frac{\partial u_i}{\partial x_j}\right| \ll 1. \tag{2.3}$$
- $\frac{\partial u_i}{\partial x_j}$ is the *displacement gradient tensor*:
$$\frac{\partial u_i}{\partial x_j} = e_{ij} + \omega_{ij}, \tag{2.4}$$
where
$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = e_{ji} \quad \text{is the } \textit{strain tensor} \text{ and} \tag{2.5}$$
$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = -\omega_{ji} \quad \text{is the } \textit{rotation tensor}. \tag{2.6}$$





- Displacements in the vicinity of $\mathbf{r}$:
$$u_i(\mathbf{r} + d\mathbf{r}) = \underbrace{u_i(\mathbf{r})}_{\substack{\text{Rigid Body}\\ \text{Translation}}} + \underbrace{\omega_{ij}dx_j}_{\substack{\text{Rigid Body}\\ \text{Rotation}}} + \underbrace{e_{ij}dx_j}_{\substack{\text{Pure}\\ \text{Deformation}}} \tag{2.7}$$

## 2.2 Rigid body rotation

- For $e_{ij} = 0$ (2.2) and (2.7):
$$d\mathbf{R} = d\mathbf{r} + \boldsymbol{\omega} \times d\mathbf{r} \tag{2.8}$$
where $\boldsymbol{\omega} = (\omega_{32}, \omega_{13}, \omega_{21})^T$. Represents rigid body rotation for $|\omega_{ij}| \ll 1$.

## 2.3 Pure deformation

### 2.3.1 Extensional deformation

- During the deformation the line element $dr_i = ds\ n_i$ is stretched to $dR_i = dS\ N_i$ ($\mathbf{n}$ and $\mathbf{N}$ are unit vectors).

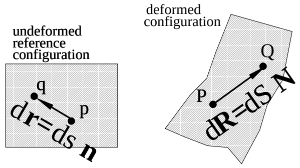


Figure 2.2: Sketch illustrating the extension (and rotation) of material line elements during the deformation of an elastic body.

- The *normal strain* $e_{\mathbf{n}}$ is the relative extension of the line element $ds\ \mathbf{n}$:
$$e_{\mathbf{n}} = \frac{dS - ds}{ds} = e_{ij}n_i n_j \tag{2.9}$$
- The $e_{(i)(i)}$ are the normal strains along the coordinate axes.

### 2.3.2 Shear deformation

- Consider the change of the angle between two material line elements $d\mathbf{r}^{(1)} = ds^{(1)}\mathbf{n}^{(1)}, d\mathbf{r}^{(2)} = ds^{(2)}\mathbf{n}^{(2)}$ which are orthogonal to each other in the undeformed state, $(dr_i^{(1)} dr_i^{(2)} = 0)$. Before the deformation: $\varphi = \pi/2$. After the deformation (see Fig. 2.3):
$$\cos\phi = 2e_{ij}n_i^{(1)}n_j^{(2)}. \tag{2.10}$$
- The $e_{ij}$ for $i \neq j$ are the *shear strains* w.r.t. the coordinate axes.

## 2.4 Principal axes/strain invariants

- The strain tensor gives the strains relative to the chosen coordinate system. Rotation of the coordinate system to a new one, such that
$$\tilde{x}_i = a_{ij}x_j \quad \text{where} \quad a_{ij}a_{kj} = \delta_{ik} \ \ (\text{orthogonal matrix, } \mathbf{A}^T = \mathbf{A}^{-1}) \tag{2.11}$$

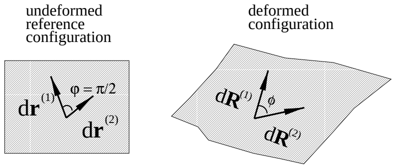


Figure 2.3: Sketch illustrating the shear deformation, i.e. the change in the angle between two material line elements during the deformation of an elastic body.

transforms the components of the strain tensor to:

$$\tilde{e_{ij}} = a_{ik} e_{kl} a_{jl} \quad \text{(symbolically } \tilde{\mathbf{E}} = \mathbf{A}\mathbf{E}\mathbf{A}^T\text{)}. \tag{2.12}$$

- There exists a special coordinate system (*principal axes*) in which $\tilde{e_{ij}} = 0$ for $i \neq j$.
- The principal axes are the normalised eigenvectors of $e_{ij}$.
- The normalised eigenvectors form the rows of the transformation matrix $a_{ik}$ to the coordinate system formed by the principal axes.
- The eigenvalues of $e_{ij}$ are the *principal strains*, i.e. the strains in the directions of the normal axes.
- The maximum normal strain, $\max e_{\mathbf{n}}$, (max. over all directions $\mathbf{n}$) is given by the maximum principal strain.
- The strain tensor has three invariants (i.e. quantities that are independent of the choice of the coordinate system):
  - **the dilation:** $d = e_{ii}$ which represents the relative change in volume
  $$d = e_{ii} = (dV - dv)/dv \tag{2.13}$$
  - **the determinant:** $\det e_{ij}$.
  - **and a third quantity:** $1/2(e_{ij}e_{ij} - e_{ii}e_{jj})$

## 2.5 Strain compatibility

- Equation (2.5) expresses $e_{ij}$ in terms of a given displacement field $u_i$.
- The inverse problem: $e_{ij}$ only describes a continuous deformation of a body (i.e. no gaps or overlaps of material develop during the deformation) iff:

$$e_{ij,kl} + e_{kl,ij} - e_{kj,il} - e_{il,kj} = 0 \tag{2.14}$$

This represents $3^4 = 81$ equations but only the ones corresponding to the following six parameter combinations are non-trivial and distinct:

| | | | | | | |
|---|---|---|---|---|---|---|
| i | 1 | 1 | 1 | 1 | 1 | 2 |
| j | 1 | 1 | 2 | 1 | 2 | 2 |
| k | 2 | 2 | 2 | 3 | 3 | 3 |
| l | 2 | 3 | 3 | 3 | 3 | 3 |

- Geometrical interpretation which motivates the derivation of eqns. (2.14): $e_{ij}$ determines the deformation of infinitesimal rectangular (cubic in 3D) blocks of material. After the deformation, the individually deformed blocks of material (deformed according to their local value of $e_{ij}$) must still fit together to form a continuous body.



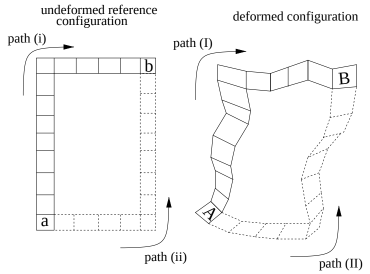


Figure 2.4: Sketch illustrating the strain compatibility condition.

## 2.6 Homogeneous deformation

- A deformation for which
$$\frac{\partial u_i}{\partial x_j} = const. \tag{2.15}$$
throughout the body is called a *homogeneous deformation*.

  Examples:

  **Simple extension** E.g. $e_{11} = e_0$, $e_{ij} = 0$ otherwise.

  **Uniform dilation** $e_{ij} = e_0 \delta_{ij}$ (spherically symmetric).

  **Simple shearing** E.g. $e_{12} = e_{21} = e_0$, $e_{ij} = 0$ otherwise.

# Chapter 3

# Analysis of stress

## 3.1 The concept of traction/stress

- If $\Delta\mathcal{F}$ is the resultant force acting on a small area element $\Delta S$ with unit normal $\mathbf{n}$, then the traction (stress) vector $\mathbf{t}$ is defined as:

$$\mathbf{t} = \lim_{\Delta S\to 0} \frac{\Delta\boldsymbol{\mathcal{F}}}{\Delta S} \tag{3.1}$$

The term 'traction' is usually used for stresses acting on the surfaces of a body.

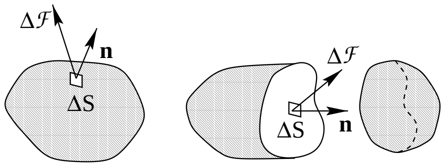


Figure 3.1: Sketch illustrating traction and stress.

## 3.2 The stress tensor

- The stress vector $\mathbf{t}$ depends on the spatial position in the body and on the orientation of the plane (characterised by the normal vector):

$$t_i = \tau_{ij} n_j, \tag{3.2}$$

where $\tau_{ij} = \tau_{ji}$ is the *stress tensor*.

- On an infinitesimal block of material whose faces are parallel to the axes, the component $\tau_{ij}$ of the stress tensor represents the traction component in the positive $i$-direction on the face $x_j = const.$ whose normal points in the positive $j$-direction (see Fig. 3.2).

## 3.3 The equations of equilibrium/motion

- The equations of equilibrium for a body, subject to a body force (force per unit volume) $F_i$ is

$$\frac{\partial \tau_{ij}}{\partial x_j} + F_i = 0. \tag{3.3}$$

- Including inertial effects via D'Alembert forces gives the equations of motion:

$$\frac{\partial \tau_{ij}}{\partial x_j} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \tag{3.4}$$





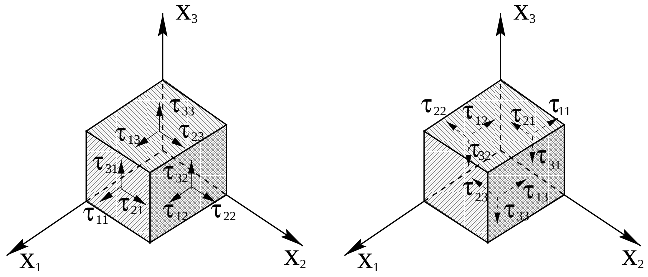


Figure 3.2: Sketch illustrating the components of the stress tensor.

where $\rho$ is the density of the body and $t$ is time.

## 3.4 Principal axes/stress invariants

- The stress tensor is real and symmetric, hence all considerations in section 2.4 apply to the stress tensor as well (transformation to different coordinate systems, principal axes, max. stress and invariants).
- In particular, we will denote the first invariant (the trace of the stress vector) by

$$\theta = \tau_{ii}. \tag{3.5}$$

## 3.5 Homogeneous stress states

- Analogous to homogeneous deformations (see section 2.6): Examples:

**Uniaxial stress** E.g. $\tau_{11} = T_0$, $\tau_{ij} = 0$ otherwise.

**Hydrostatic pressure** $\tau_{ij} = P_0 \delta_{ij}$ (spherically symmetric).

**Pure shear stress** E.g. $\tau_{12} = \tau_{21} = T_0$, $\tau_{ij} = 0$ otherwise.

# Chapter 4

# Elasticity & constitutive equations

## 4.1 The constitutive equations

- The constitutive equations determine the stress $\tau_{ij}$ in the body as function of the body's deformation.

**Definition:** A solid body is called *elastic* if

$$\tau_{ij}(x_n, t) = \tau_{ij}(e_{kl}(x_n, t)). \tag{4.1}$$

i.e. if the stress depends on the *instantaneous, local* values of the strain only.

- For *small* strains, a Taylor expansion of (4.1) gives:

$$\tau_{ij} = \underbrace{\left.\tau_{ij}\right|_{e_{kl}=0}}_{\text{Initial Stress } \tau^0_{ij}} + \underbrace{\left.\frac{\partial \tau_{ij}}{\partial e_{kl}}\right|_{e_{kl}=0}}_{E_{ijkl}} e_{kl}. \tag{4.2}$$

- If the reference configuration coincides with a stress free state, then $\tau^0_{ij} = 0$ and we obtain *Hooke's law*:

$$\tau_{ij} = E_{ijkl} e_{kl}. \tag{4.3}$$

**Definition:** A solid body is called *homogeneous* if $E_{ijkl}$ is independent of $x_i$.

**Definition:** A solid body is called *isotropic* if its elastic properties are the same in all directions.

- For an isotropic homogeneous elastic solid:

$$E_{ijkl} = \lambda \delta_{ij} \delta_{kl} + 2\mu \delta_{ik} \delta_{jl}, \tag{4.4}$$

where $\lambda$ and $\mu$ are the *Lamé constants.*

- Stress-strain relationship for an isotropic homogeneous elastic solid:

$$\tau_{ij} = \lambda \delta_{ij} \underbrace{e_{kk}}_{=d} + 2\mu e_{ij}, \tag{4.5}$$

and in the inverse form:

$$e_{ij} = \underbrace{\frac{1}{2\mu}\left(\delta_{ik}\delta_{jl} - \frac{\lambda}{(3\lambda + 2\mu)}\delta_{ij}\delta_{kl}\right)}_{D_{ijkl}} \tau_{kl} \tag{4.6}$$

so that

$$e_{ij} = D_{ijkl} \tau_{kl}. \tag{4.7}$$

Written out:

$$e_{ij} = \frac{1}{2\mu}\tau_{ij} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\delta_{ij} \underbrace{\tau_{kk}}_{=\theta} \tag{4.8}$$





- For an isotropic homogeneous elastic solid the principal axes of the stress and strain tensors coincide and

$$\theta = \tau_{kk} = (3\lambda + 2\mu) d = (3\lambda + 2\mu) e_{kk} \tag{4.9}$$

## 4.2 Experimental determination of elastic constants

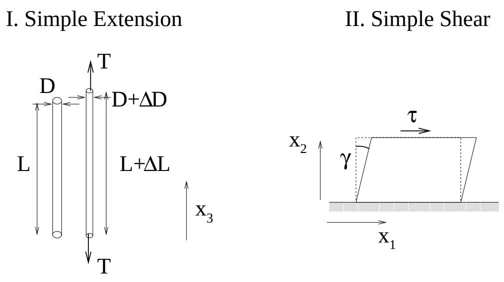


Figure 4.1: Sketch illustrating the two fundamental experiments for the determination of the elastic constants.

### 4.2.1 Experiment I: Simple extension of a thin cylinder

- Observations:

$$T = EA\frac{\Delta L}{L} \tag{4.10}$$

i.e.

$$\tau_{33} = E e_{33} \tag{4.11}$$

(since $e_{33} = \Delta L / L$) and

$$\frac{e_{11}}{e_{33}} = \frac{e_{22}}{e_{33}} = -\nu \tag{4.12}$$

where $e_{11} = e_{22} = \Delta D / D$.

- $E$ and $\nu$ are *Young's modulus* and *Poisson's ration*, respectively.

### 4.2.2 Experiment II: Simple shear

- Observation:

$$\tau = G\gamma \tag{4.13}$$

i.e.

$$\tau_{12} = G \; 2e_{12}. \tag{4.14}$$

- $G$ is the material's *shear modulus.*

### 4.2.3 Constitutive equations in terms of $E$ and $\nu$

$$\tau_{ij} = \frac{E}{1+\nu}\left(e_{ij} + \frac{\nu}{1-2\nu}\delta_{ij} \underbrace{e_{kk}}_{d}\right). \tag{4.15}$$

- Note that materials with $\nu = 1/2$ are *incompressible*, i.e. $d \equiv 0$.

$$e_{ij} = \frac{1}{E}\left((1+\nu)\tau_{ij} - \nu \delta_{ij} \underbrace{\tau_{kk}}_{\theta}\right). \tag{4.16}$$

### 4.3 Relations between the elastic constants

| | $\lambda =$ | $\mu = G =$ | $E =$ | $\nu =$ |
|---|---|---|---|---|
| $\lambda, \mu$ | $\lambda$ | $\mu$ | $\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ | $\frac{\lambda}{2(\lambda+\mu)}$ |
| $\lambda, \nu$ | $\lambda$ | $\frac{\lambda(1-2\nu)}{2\nu}$ | $\frac{(1+\nu)(1-2\nu)\lambda}{\nu}$ | $\nu$ |
| $\mu, E$ | $\frac{\mu(E-2\mu)}{3\mu-E}$ | $\mu$ | $E$ | $\frac{E-2\mu}{2\mu}$ |
| $E, \nu$ | $\frac{E\nu}{(1+\nu)(1-2\nu)}$ | $\frac{E}{2(1+\nu)}$ | $E$ | $\nu$ |

# Chapter 5

# The equations of linear elasticity

## 5.1 Summary of equations

- Strain-displacement relations:
$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{5.1}$$
- Equilibrium equations/equations of motion:
$$\tau_{ij,j} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{5.2}$$
- Constitutive equations:
$$\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} \tag{5.3}$$

## 5.2 Displacement formulation: The Navier-Lamé equations

- Solve for the displacements:
$$(\lambda + \mu) u_{k,ki} + \mu u_{i,kk} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{5.4}$$
or symbolically:
$$(\lambda + \mu) \text{ grad div } \mathbf{u} + \mu \nabla^2 \mathbf{u} + \mathbf{F} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \tag{5.5}$$
which is equivalent to:
$$(\lambda + 2\mu) \text{ grad div } \mathbf{u} - \mu \text{ curl curl } \mathbf{u} + \mathbf{F} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{5.6}$$
- This is a system of three coupled linear elliptic PDEs for the three displacements $u_i(x_j)$.

## 5.3 Stress formulation: The static Beltrami-Mitchell equations

- For static deformations, we have
$$\frac{1-\nu}{1+\nu} \underbrace{\tau_{ii,jj}}_{\theta_{,jj}} + F_{i,i} = 0 \quad \text{or symbolically} \quad \frac{1-\nu}{1+\nu} \nabla^2 \theta + \text{div } \mathbf{F} = 0. \tag{5.7}$$
and the stresses fulfil the Beltrami-Mitchell equations:
$$\underbrace{\tau_{ij,kk}}_{\nabla^2 \tau_{ij}} + \frac{1}{1+\nu} \underbrace{\tau_{kk,ij}}_{\theta_{,ij}} + \frac{\nu}{1-\nu} \delta_{ij} \underbrace{F_{k,k}}_{\text{div } \mathbf{F}} + F_{j,i} + F_{i,j} = 0. \tag{5.8}$$
- (5.8) represents a system of six coupled linear elliptic PDEs for the six stress components $\tau_{ij}(x_j)$. When these have been determined, the strains can be recovered from (4.6) or (4.16). Then the displacements follow from (5.1). They are only determined up to arbitrary rigid body motions.

## 5.4 Simplifications for $\mathbf{F} = const.$:

- For constant (or vanishing!) body force, the stress, strain and displacement components are biharmonic functions,
$$u_{i,jjkk} = 0 \qquad \tau_{ij,kkll} = 0 \qquad e_{ij,kkll} = 0 \tag{5.9}$$
or symbolically:
$$\nabla^4 \mathbf{u} = 0 \qquad \nabla^4 \tau_{ij} = 0 \qquad \nabla^4 e_{ij} = 0. \tag{5.10}$$

- The dilation and the trace of the stress tensor are harmonic functions:
$$u_{j,jkk} = d_{,kk} = 0 \qquad \tau_{jj,kk} = \theta_{,kk} = 0 \tag{5.11}$$
or symbolically:
$$\nabla^2 d = 0 \qquad \nabla^2 \theta = 0 \tag{5.12}$$

- Note that in (5.4) – (5.8) $\mathbf{F}$ acts as an inhomogeneity in a system of linear equations. The system can be transformed into a homogeneous system for $\mathbf{u}_h = \mathbf{u} - \mathbf{u}_p$ (with different boundary conditions) if a particular solution $\mathbf{u}_p$ (which does not have to fulfil the boundary conditions) can be found.

## 5.5 Boundary conditions:

- Displacement (Dirichlet) boundary conditions: Prescribed displacement field $u_i^{(0)}$.
$$u_i|_{\partial D} = u_i^{(0)} \tag{5.13}$$

- Stress (Neumann) boundary conditions: Prescribed (applied) traction $t_i^{(0)}$ on boundary. Note that $n_j$ is the *outer* unit normal vector on the elastic body.
$$t_i|_{\partial D} = \tau_{ij} n_j|_{\partial D} = t_i^{(0)} \tag{5.14}$$

- Mixed (Robin) boundary conditions – 'elastic foundation' represented by the stiffness tensor $k_{ij}$. Physically, this implies that the traction which the elastic foundation exerts on the body is proportional to the boundary displacement. This can be combined with an applied traction $t_i^{(0)}$ as in the Neumann case.
$$(t_i + k_{ij} u_j)\,|_{\partial D} = (\tau_{ij} n_j + k_{ij} u_j)\,|_{\partial D} = t_i^{(0)} \tag{5.15}$$



### Governing Equations in Cylindrical Polar Coordinates

- $x_1 = x = r\cos\theta,\ x_2 = y = r\sin\theta,\ x_3 = z = z.$
$$\mathbf{u} = (u_r, u_\theta, u_z), \quad e = (e_{ij}), \quad \tau = (\tau_{ij}), \quad \text{where } i,j = r,\theta,z.$$

- Vector calculus:
$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, \qquad \operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$
$$\operatorname{curl}\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{z}}.$$

- Stress-strain relations have the same form as in Cartesian coordinates:
$$\tau_{ij} = \lambda \delta_{ij} \operatorname{div}\mathbf{u} + 2\mu e_{ij}, \qquad i,j = r,\theta,z.$$

- Stress-displacement relations:
$$\tau_{rr} = \lambda \operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda \operatorname{div}\mathbf{u} + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right), \quad \tau_{zz} = \lambda \operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_z}{\partial z},$$
$$\frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad \frac{\tau_{rz}}{\mu} = \frac{\tau_{zr}}{\mu} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \quad \frac{\tau_{\theta z}}{\mu} = \frac{\tau_{z\theta}}{\mu} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

- Strain-displacement relations:
$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \qquad e_{zz} = \frac{\partial u_z}{\partial z},$$
$$2e_{r\theta} = 2e_{\theta r} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad 2e_{rz} = 2e_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \quad 2e_{z\theta} = 2e_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

- Equilibrium equations (statics): for the displacement formulation, use Navier's equation,
$$(\lambda + 2\mu)\operatorname{grad}\operatorname{div}\mathbf{u} - \mu \operatorname{curl}\operatorname{curl}\mathbf{u} + \mathbf{F} = \mathbf{0},$$
whereas for the stress formulation, use
$$\begin{aligned}
\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} + F_r &= 0 \\
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} + F_\theta &= 0 \\
\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r}\tau_{rz} + F_z &= 0.
\end{aligned}$$

- Stress boundary conditions: these are when $\mathbf{T}$ is prescribed. We have, from $T_i = \hat{n}_j \tau_{ij}$,
$$\begin{aligned}
T_r &= \hat{n}_r \tau_{rr} + \hat{n}_\theta \tau_{r\theta} + \hat{n}_z \tau_{rz} \\
T_\theta &= \hat{n}_r \tau_{r\theta} + \hat{n}_\theta \tau_{\theta\theta} + \hat{n}_z \tau_{\theta z} \\
T_z &= \hat{n}_r \tau_{rz} + \hat{n}_\theta \tau_{\theta z} + \hat{n}_z \tau_{zz}
\end{aligned}$$

### Governing Equations in Spherical Polar Coordinates

- $x_1 = x = r\sin\theta\cos\phi$, $x_2 = y = r\sin\theta\sin\phi$, $x_3 = z = r\cos\theta$.

$$\mathbf{u} = (u_r, u_\theta, u_\phi), \quad e = (e_{ij}), \quad \tau = (\tau_{ij}), \quad \text{where } i,j = r,\theta,\phi.$$

- Vector calculus:

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}},$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta\, u_r) + \frac{\partial}{\partial\theta}(r\sin\theta\, u_\theta) + \frac{\partial}{\partial\phi}(r u_\phi)\right\},$$

$$\operatorname{curl}\mathbf{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ u_r & r u_\theta & r\sin\phi\, u_\phi \end{vmatrix}.$$

- Stress-strain relations have the same form as in Cartesian coordinates:

$$\tau_{ij} = \lambda\delta_{ij}\operatorname{div}\mathbf{u} + 2\mu e_{ij}, \qquad i,j = r,\theta,\phi.$$

- Stress-displacement relations:

$$\tau_{rr} = \lambda\operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda\operatorname{div}\mathbf{u} + \frac{2\mu}{r}\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right),$$

$$\tau_{\phi\phi} = \lambda\operatorname{div}\mathbf{u} + \frac{2\mu}{r}\left(\frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial\phi} + u_r + u_\theta\cot\theta\right), \quad \frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta},$$

$$\frac{\tau_{r\phi}}{\mu} = \frac{\tau_{\phi r}}{\mu} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}, \quad \frac{\tau_{\theta\phi}}{\mu} = \frac{\tau_{\phi\theta}}{\mu} = \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial\theta} - \frac{u_\phi\cot\theta}{r}.$$

- Strain-displacement relations:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \qquad e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r},$$

$$2e_{r\theta} = 2e_{\theta r} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}, \quad 2e_{r\phi} = 2e_{\phi r} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r},$$

$$2e_{\phi\theta} = 2e_{\theta\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial\theta} - \frac{u_\phi\cot\theta}{r}.$$

- Equilibrium equations (statics): for the displacement formulation, use Navier's equation,

$$(\lambda + 2\mu)\operatorname{grad}\operatorname{div}\mathbf{u} - \mu\operatorname{curl}\operatorname{curl}\mathbf{u} + \mathbf{F} = \mathbf{0},$$

whereas for the stress formulation, use

$$\begin{aligned}
\frac{\partial\tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{r\phi}}{\partial\phi} + \frac{2\tau_{rr} - \tau_{\theta\theta} - \tau_{\phi\phi} + \cot\theta\,\tau_{r\theta}}{r} + F_r &= 0 \\
\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{3\tau_{r\theta} + (\tau_{\theta\theta} - \tau_{\phi\phi})\cot\theta}{r} + F_\theta &= 0 \\
\frac{\partial\tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi\phi}}{\partial\phi} + \frac{3\tau_{r\phi} + 2\tau_{\theta\phi}\cot\theta}{r} + F_\phi &= 0.
\end{aligned}$$

- Stress boundary conditions: these are when $\mathbf{T}$ is prescribed. We have, from $T_i = \hat{n}_j\tau_{ij}$,

$$\begin{aligned}
T_r &= \hat{n}_r\tau_{rr} + \hat{n}_\theta\tau_{r\theta} + \hat{n}_\phi\tau_{r\phi} \\
T_\theta &= \hat{n}_r\tau_{r\theta} + \hat{n}_\theta\tau_{\theta\theta} + \hat{n}_\phi\tau_{\theta\phi} \\
T_\phi &= \hat{n}_r\tau_{r\phi} + \hat{n}_\theta\tau_{\theta\phi} + \hat{n}_\phi\tau_{\phi\phi}
\end{aligned}$$

# Chapter 6

# Plane strain problems

## 6.1 Basic equations

**Definition:** A deformation is said to be one of *plane strain* (parallel to the plane $x_3 = 0$) if:

$$u_3 = 0 \quad \text{and} \quad u_\alpha = u_\alpha(x_\beta). \tag{6.1}$$

- There are only two independent variables, $(x_1, x_2) = (x, y)$.
- Plane strain is only possible if $F_3 = 0$.
- Only the in-plane strains are non-zero, $e_{i3} = 0$.
- Stress-strain relationship:

$$\tau_{\alpha\beta} = \lambda\delta_{\alpha\beta}e_{\gamma\gamma} + 2\mu e_{\alpha\beta}. \tag{6.2}$$

$$2\mu e_{\alpha\beta} = \tau_{\alpha\beta} - \nu\delta_{\alpha\beta}\underbrace{\tau_{\gamma\gamma}}_{\tilde{\theta}} \tag{6.3}$$

$$\tau_{33} = \nu\tau_{\gamma\gamma} = \nu\tilde{\theta} \tag{6.4}$$

- Static equilibrium equations:

$$\tau_{\alpha\beta,\beta} + F_\alpha = 0 \tag{6.5}$$

- Compatibility equation: Only one non-trivial equation

$$0 = e_{11,22} + e_{22,11} - 2e_{12,12} \tag{6.6}$$

Formulated in terms of stresses:

$$(1-\nu)\tilde{\theta}_{,\alpha\alpha} + F_{\alpha,\alpha} = 0, \tag{6.7}$$

or symbolically

$$(1-\nu)\tilde{\nabla}^2\tilde{\theta} + \operatorname{div}\mathbf{F} = 0, \tag{6.8}$$

where $\tilde{\nabla}^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

## 6.2 The Airy stress function

- For $\mathbf{F} = 0$ the in-plane stresses can be expressed in terms of the *Airy stress function* $\phi$:

$$\tau_{11} = \frac{\partial^2\phi}{\partial y^2}, \quad \tau_{22} = \frac{\partial^2\phi}{\partial x^2}, \quad \tau_{12} = -\frac{\partial^2\phi}{\partial x\partial y}. \tag{6.9}$$

- The Airy stress function is *biharmonic*:

$$\tilde{\nabla}^4\phi = \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0. \tag{6.10}$$

## 6.3 The stress boundary conditions in terms of the Airy stress function

- The applied tractions along the boundary $\partial D$ (parametrised by the arclength $s$) are given in terms of the Airy stress function $\phi$ by
$$t_1(s) = t_x(s) = \frac{d}{ds}\left(\frac{\partial\phi}{\partial y}\right) \tag{6.11}$$
and
$$t_2(s) = t_y(s) = -\frac{d}{ds}\left(\frac{\partial\phi}{\partial x}\right). \tag{6.12}$$

- Hence, if $t_\alpha(s)$ is given, the boundary conditions for $\phi$ can be derived by the following procedure:
  1. Integrate (6.11) and (6.12) along the boundary (w.r.t. $s$). This provides $(\partial\phi/\partial x, \partial\phi/\partial y)^T = \tilde\nabla\phi$ on the boundary.
  2. Rewrite $\tilde\nabla\phi = \partial\phi/\partial s\ \mathbf{e}_t + \partial\phi/\partial n\ \mathbf{e}_n$ where $\mathbf{e}_t$ and $\mathbf{e}_n$ are the unit tangent and (outer) normal vectors on the boundary. This provides $\partial\phi/\partial s$ and $\partial\phi/\partial n$ along the boundary.
  3. Integrate $\partial\phi/\partial s$ along the boundary (w.r.t. $s$). This provides $\phi$ along the boundary.

- After this procedure $\phi$ and $\partial\phi/\partial n$ are known along the entire boundary and can be used as the boundary condition for the fourth order biharmonic equation (6.10).

- Note: any constants of integration arising during the procedure can be set to zero.

- For a traction free boundary, $t_\alpha(s) = 0$, we can use the boundary conditions:
$$\phi = 0 \quad \text{and} \quad \partial\phi/\partial n = 0 \quad \text{on } \partial D \tag{6.13}$$

## 6.4 The displacements in terms of the Airy stress function

- For a given Airy stress function $\phi(x,y)$, the displacements $u(x,y), v(x,y)$, are determined by
$$2\mu\frac{\partial u}{\partial x} = (1-\nu)\tilde\nabla^2\phi - \frac{\partial^2\phi}{\partial x^2}, \tag{6.14}$$
$$2\mu\frac{\partial v}{\partial y} = (1-\nu)\tilde\nabla^2\phi - \frac{\partial^2\phi}{\partial y^2} \tag{6.15}$$
and
$$\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -\frac{\partial^2\phi}{\partial x\partial y}. \tag{6.16}$$

- One way to determine the displacement fields from these equations is given by the following procedure:
  1. Get $p(x,y) = \tilde\nabla^2\phi(x,y)$ from the known $\phi(x,y)$.
  2. $p(x,y)$ is a harmonic function; determine its complex conjugate $q(x,y)$ from the Cauchy-Riemann equations:
$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} \quad \text{and} \quad \frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x}. \tag{6.17}$$
  3. Integrate $f(z) = f(x+iy) = p(x,y) + i\ p(x,y)$ and thus determine $P(x,y)$ and $Q(x,y)$ from
$$F(z) = \int f(z)dz =: P(x,y) + i\ Q(x,y). \tag{6.18}$$
  4. Then the displacements are given by:
$$u(x,y) = \frac{1}{2\mu}\Big[(1-\nu)\ P(x,y) - \frac{\partial\phi}{\partial x} + \underbrace{a+cy}_{\text{rigid body motion}}\Big] \tag{6.19}$$
and
$$v(x,y) = \frac{1}{2\mu}\Big[(1-\nu)\ Q(x,y) - \frac{\partial\phi}{\partial y} + \underbrace{b-cx}_{\text{rigid body motion}}\Big]. \tag{6.20}$$



## 6.5 Equations in polar coordinates

- The biharmonic equation in polar coordinates:
$$\tilde\nabla^4\phi(r,\varphi) = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right]\left[\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2}\right] \tag{6.21}$$
$$\tilde\nabla^4\phi(r,\varphi) = \phi_{,rrrr} + \frac{2}{r}\phi_{,rrr} - \frac{1}{r^2}(\phi_{,rr} - 2\phi_{,rr\varphi\varphi}) + \frac{1}{r^3}(\phi_{,r} - 2\phi_{,r\varphi\varphi}) + \frac{1}{r^4}(4\phi_{,\varphi\varphi} + 2\phi_{,\varphi\varphi\varphi\varphi}) \tag{6.22}$$

- For axisymmetric solutions:
$$\tilde\nabla^4\phi(r) = \frac{1}{r}\left[r\left(\frac{1}{r}[r\phi_{,r}]_{,r}\right)_{,r}\right]_{,r} \tag{6.23}$$
$$\tilde\nabla^4\phi(r) = \phi_{,rrrr} + \frac{2}{r}\phi_{,rrr} - \frac{1}{r^2}\phi_{,rr} + \frac{1}{r^3}\phi_{,r} \tag{6.24}$$

- Stresses:
$$\tau_{rr} = \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}, \tag{6.25}$$
$$\tau_{\varphi\varphi} = \frac{\partial^2\phi}{\partial r^2} \tag{6.26}$$
$$\tau_{r\varphi} = \frac{1}{r^2}\frac{\partial\phi}{\partial\varphi} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\varphi} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\varphi}\right). \tag{6.27}$$

## 6.6 Particular solutions of the biharmonic equation

### 6.6.1 Harmonic functions

- Obviously, all harmonic functions also fulfil the biharmonic equation.

### 6.6.2 Power series expansions

$$\phi(x,y) = \sum_{i,k} a_{ik}x^i y^k \tag{6.28}$$

- Any terms with $i+k<2$ do not give a contribution.
- Any terms with $i+k<4$ fulfil $\tilde\nabla^4\phi = 0$ for arbitrary constants $a_{ik}$. Special cases are:

| $\phi(x,y)$ | $\tau_{xx}$ | $\tau_{yy}$ | $\tau_{xy}$ | Interpretation: |
|---|---|---|---|---|
| $a_{02}\ y^2$ | $2\ a_{02}$ | 0 | 0 | constant tension in x-direction |
| $a_{11}\ xy$ | 0 | 0 | $-a_{11}$ | pure shear |
| $a_{20}\ x^2$ | 0 | $2\ a_{20}$ | 0 | constant tension in y-direction |
| $a_{03}\ y^3$ | $6\ a_{03}y$ | 0 | 0 | pure x-bending |
| $a_{30}\ x^3$ | 0 | $6\ a_{30}x$ | 0 | pure y-bending |

- Linear combinations provide stress fields for combined load cases.

### 6.6.3 Solutions in polar coordinates

- The general axisymmetric solution:

$$\phi(r) = A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r \tag{6.29}$$

- The general separated non-axisymmetric solution:

**For $n = 1$:**

$$\begin{aligned}\phi(r,\varphi) = &\left(Ar + \frac{B}{r} + Cr^3 + Dr\log r\right) \quad \cos(\varphi)\\ &+ \left(ar + \frac{b}{r} + cr^3 + dr\log r\right) \quad \sin(\varphi)\end{aligned} \tag{6.30}$$

**For $n \geq 2$:**

$$\begin{aligned}\phi(r,\varphi) = \sum_{n=2}^{\infty} &\left(A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2}\right) \quad \cos(n\varphi)\\ &+ \left(a_n r^n + b_n r^{-n} + c_n r^{n+2} + d_n r^{-n+2}\right) \quad \sin(n\varphi)\end{aligned} \tag{6.31}$$

## 6.7 St. Venant's principle

Section 6.6 provides many solutions of the biharmonic equation. The free constants in these solutions have to be determined from the boundary conditions. This is the hardest part of the solution! 'Good' approximate solutions can often be obtained by using:

> *St. Venant's principle*
>
> In elastostatics, if the boundary tractions $\mathbf{t}$ on a part $\partial D_1$ of the boundary $\partial D$ are replaced by a statically equivalent traction distribution $\hat{\mathbf{t}}$, the effects on the stress distribution in the body are negligible at points whose distance from $\partial D_1$ is large compared to the maximum distance between the points of $\partial D_1$.

*'Statically equivalent'* means that the resultant forces and moments due to the two tractions $\mathbf{t}$ and $\hat{\mathbf{t}}$ are identical. Hence, the traction boundary conditions are not fulfilled pointwise but in an average sense.