LECTURE 9

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Subject: RE: homogeneous
From: Ruth Thomas <ruth.thomas@manchester.ac.uk>
Date: 19/10/12 08:58
To: Matthias Heil <M.Heil@maths.man.ac.uk>

Dear Matthias,
If you wish, though I can't imagine why you would want to!.
Kind regards,
Ruth

Dr. Ruth M. Thomas
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From: Matthias Heil [M.Heil@maths.manchester.ac.uk]
Sent: 18 October 2012 17:51
To: Ruth Thomas
Subject: homogeneous

Ruth,

can we have a quick emergency meeting about the definition of homogeneous?

Thanks,

Matthias

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Professor Matthias Heil

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U.K.

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"Homogeneous" and all that...

Fact:

- Even though mathematics is a (very!) exact science, it is virtually unavoidable that certain terms are used with slightly different definitions in different contexts.
- Worse: Sometimes different people use the same term with a slightly different meaning in the same context.
- This does not cause any problems, provided it is clear what is meant!
- $\Longrightarrow$ Clashes between different lecturers/authors/... are unavoidable.
- $\Longrightarrow$ Clashes within a single course are avoidable, but...

Mid-term test / Exam:

Given the enormous confusion (for which I apologise on behalf of Dave Harris) caused by the multiple different definitions of the above term in the printed lecture notes and the example sheet questions, we have decided the following:

- The definitions of homogenous/inhomogenous/... will NOT feature in any of these assessments.
- The mid-term test will only cover material up to (but not including) the method of characteristics.
- I will be closely involved in the setting of the exam and the mid-term test to ensure that the material that I taught is examined in a way that is consistent with the way it was presented in the lecture.

Example:

$$\underbrace{x}_{a}\frac{\partial u}{\partial x} + \underbrace{y}_{b}\frac{\partial u}{\partial y} = \underbrace{2xy}_{c} \quad \text{(linear)}$$

1 Characteristics:

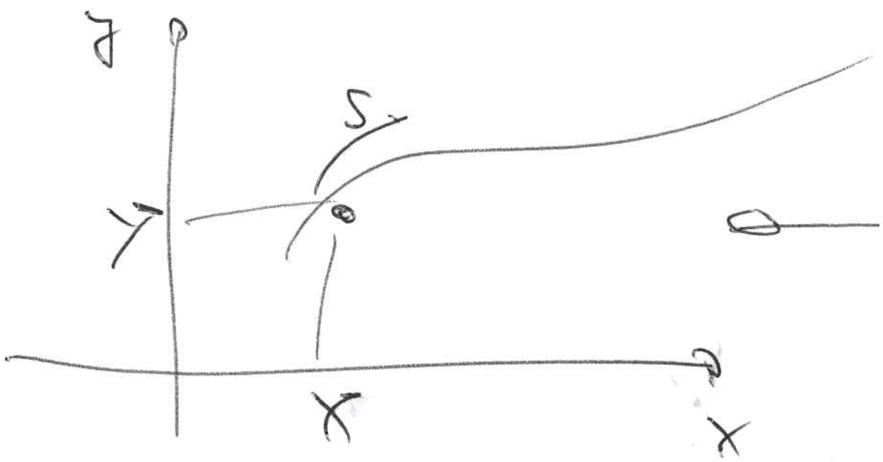
$$\left.\begin{array}{l} \boxed{\dfrac{dx}{ds} = a(x,y) = x} \\ \boxed{\dfrac{dy}{ds} = b(x,y) = y} \end{array}\right\}$$

System of two 1st order ODEs for $x(s)$ & $y(s)$

$$\to x(s) = X e^{s}$$
$$\to y(s) = Y e^{s}$$

where X & Y are constants (w.r.t. s)

Notation correct because $x(s=0) = X$ & $y(s=0) = Y$

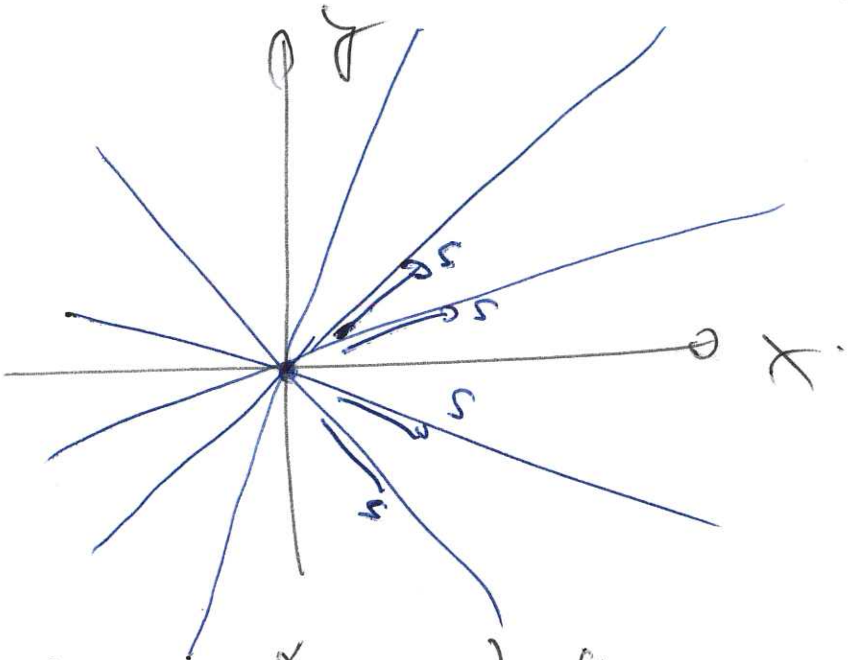


← generic picture.

In our specific example:

Note $\dfrac{x(s)}{y(s)} = \dfrac{X \cancel{e^{s}}}{Y \cancel{e^{s}}} = \dfrac{X}{Y}$ = const. along charact.

$\Rightarrow$ characteristics are straight lines through the origin. (4



② along (any of) these charact:

$$\frac{du}{ds} = c(x, y, u)$$

Here: $\frac{du}{ds} = 2xy$ on charact!

$$= 2x(s)\,y(s)$$

$$= 2\underbrace{Xe^{s}}_{x(s)}\,\underbrace{Ye^{s}}_{y(s)}$$

$$\frac{du}{ds} = 2XYe^{2s}$$ 1st order ODE for $u(s)$

$$u(s) = XYe^{2s} + C$$

This is u(s) along the

characteristic line that goes through $(\bar{x}, \bar{y})$. C is the constant of integration - it is constant along that characteristic. Its value is determined by the IC for u.

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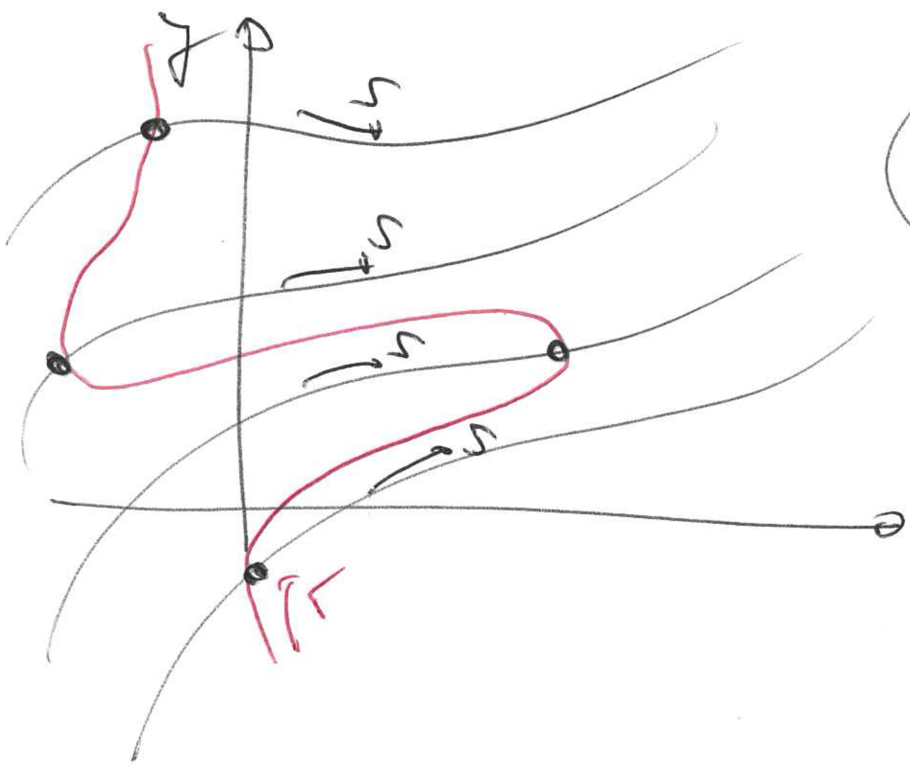
(3) IC: Recall: can/must provide initial conditions for u along a line that intersects all the characteristics once.



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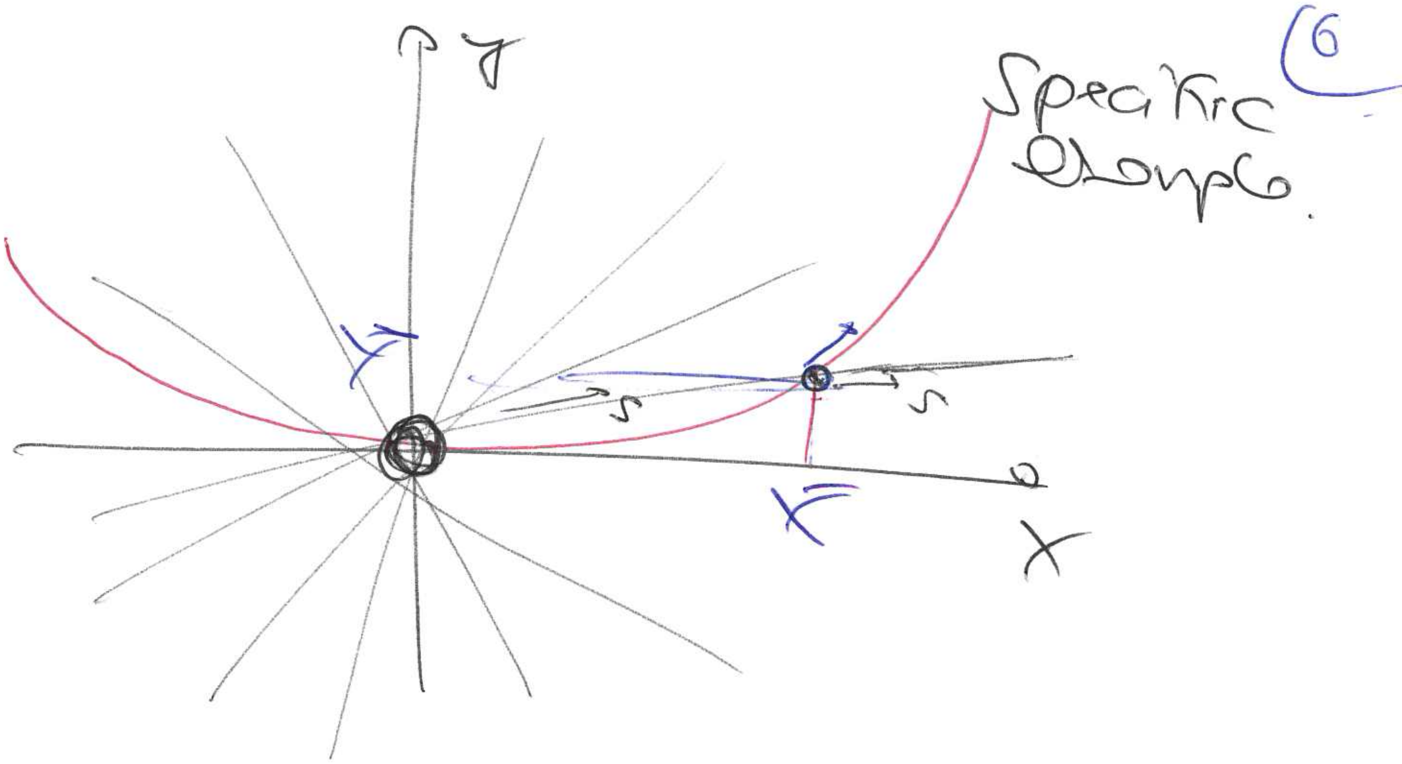

we choose: $u = 1$ along the line $y = 2x^2$.

Parametise the line along which the IC is applied.

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \bar{X}(r) \\ \bar{Y}(r) \end{pmatrix} = \begin{pmatrix} r \\ 2r^2 \end{pmatrix}$$

$$u(r, s=0) = 1 = \bar{X}(r)\,\bar{Y}(r)\,\underbrace{e^{2s}}_{1} + C'(r)$$

$$1 = r\,2r^2 + C'(r)$$

$$C'(r) = 1 - 2r^3$$

parametric soln:

$$u(r,s) = \bar{X}(r)\, y(r)\, e^{2s} + C'(r)$$

$$= 2r^3 e^{2s} + (1 - 2r^3)$$

$$\boxed{u(r,s) = 1 + 2r^3 (e^{2s} - 1)}$$

for
$$\boxed{\begin{aligned} x(r,s) &= \bar{X}(r) e^{s} = r e^{s} \\ y(r,s) &= \bar{Y}(r) e^{s} = 2r^2 e^{s} \end{aligned}}$$

Given $(r,s) \Rightarrow x(r,s), y(r,s) \Rightarrow u$

To turn this parametric form into $u(x,y)$: Invert the mapping between $(x,y) \Leftrightarrow (r,s)$