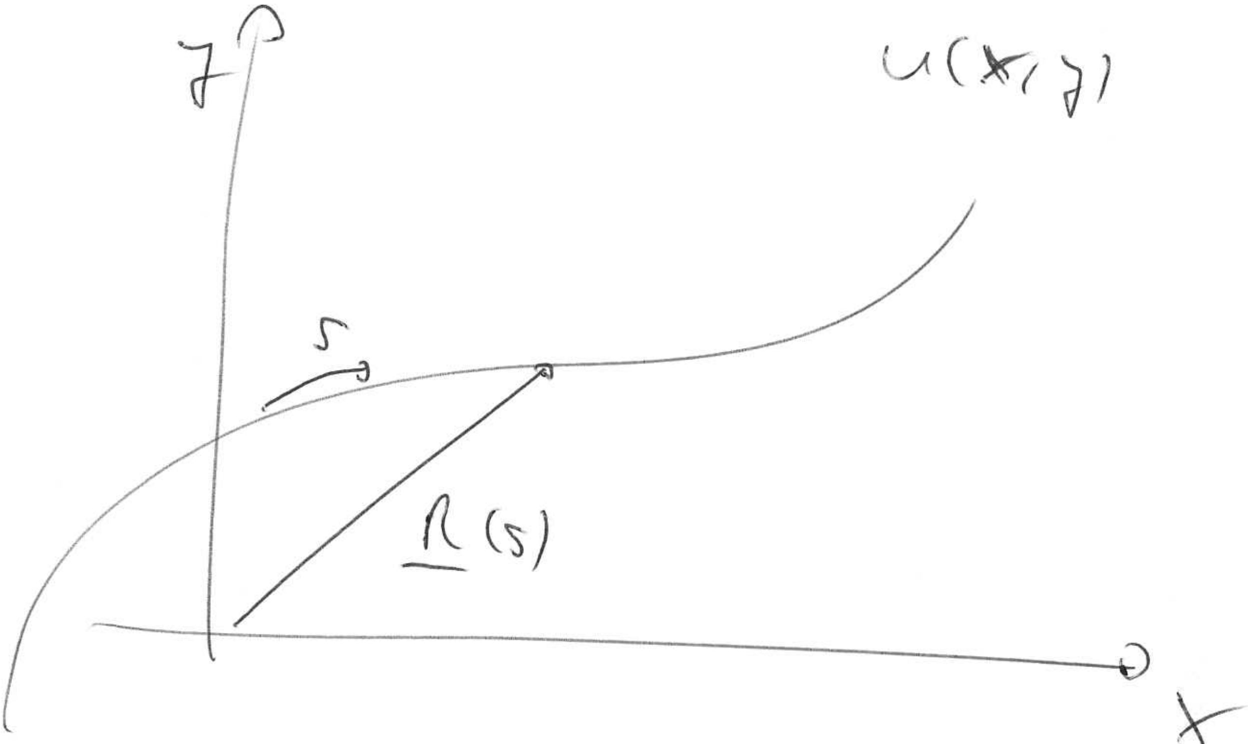
LECTURE 8

Method of characteristics

1

$$a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}=c(x,y,u)$$

recall path derivative



rate of change of u along that path $\underline{R}(s)=\begin{pmatrix}x(s)\\y(s)\end{pmatrix}$

then:

$$\frac{du}{ds}=\frac{dx}{ds}\frac{\partial u}{\partial x}+\frac{dy}{ds}\frac{\partial u}{\partial y}$$

Strategy

(1) choose a path

$$\boxed{\frac{dx}{ds}=a(x,y)} \quad \& \quad \boxed{\frac{dy}{ds}=b(x,y)}$$

i.e. a path that satisfies these two coupled 1st order ODEs subject to ICs:

$$x(s=0) = X$$
$$y(s=0) = Y$$

(2) Along this path

$$\underbrace{a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y}}_{} = c(x,y,u)$$

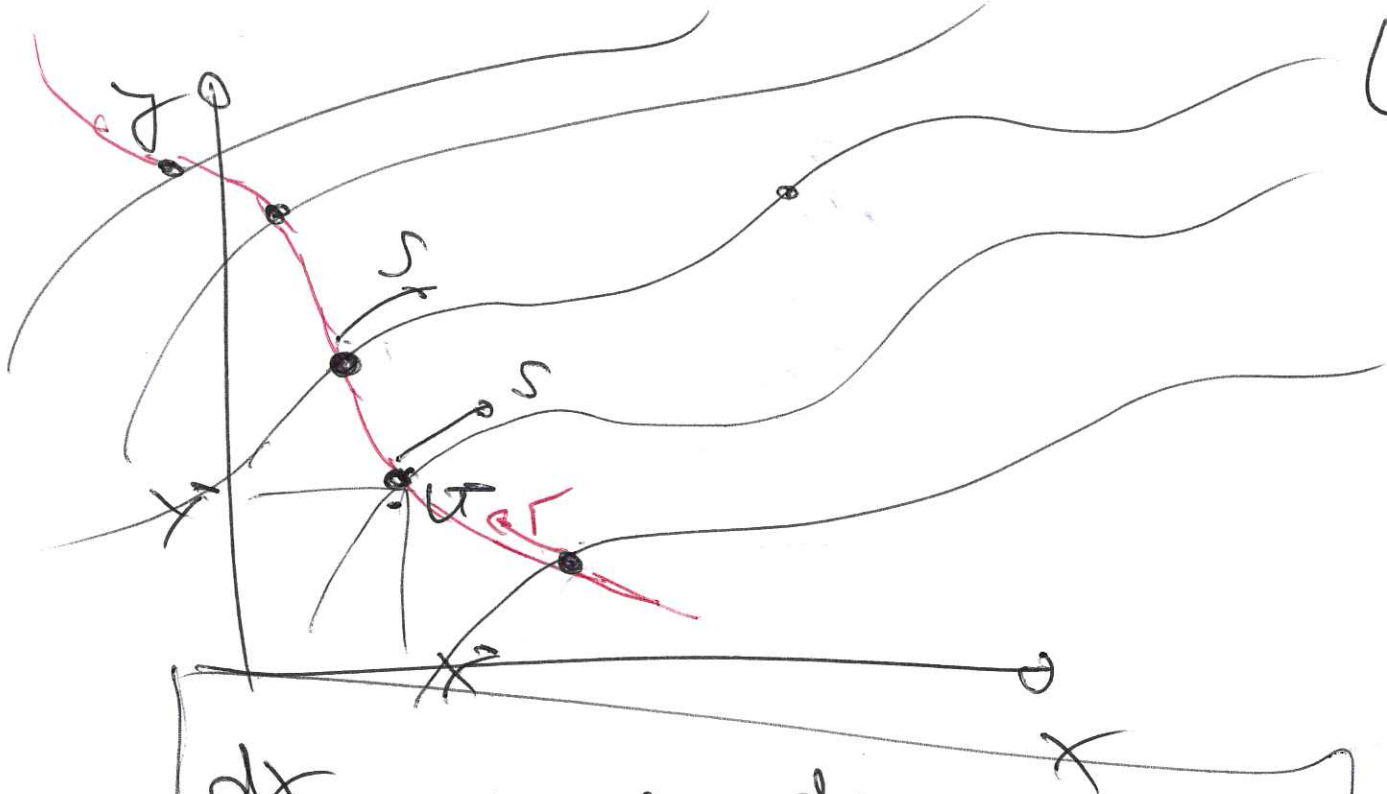
$$\frac{du}{ds} = c(x,y,u)$$

$$\frac{du}{ds} = c\left(x(s), y(s), u(s)\right)$$

Another 1st order ODE for $u(s)$ subj to some IC

$$u(s=0) = U.$$

Build up the soln via families of so-called characteristic lines $(x(s), y(s))$ (characteristics).



$$\boxed{\frac{dx}{ds} = a(x,y) \qquad \frac{dy}{ds} = b(x,y) \qquad x(s=0) = X \qquad y(s=0) = Y}$$

Then integrate

$$\boxed{\frac{du}{ds} = c(x(s), y(s), u(s)) \quad \text{subj to } u(s=0) = U}$$

Note: We can build up the soln everywhere if

(1) IC is specified along a line that crosses all the characteristics.

(2) Characteristics themselves ~~have~~ ~~to~~ sweep out the entire x-y plane.

Systematic construction of soln:

1. Determine char. curves $x(s), y(s)$ for some IC s.th. $x(s=0) = \bar{x}$ $y(s=0) = \bar{y}$

2. Specify the soln along some initial line
$$\begin{pmatrix} \bar{x}(r) \\ \bar{y}(r) \end{pmatrix} \quad \& \quad \bar{u}(r)$$

3. integrate the soln along the characteristics
$$\frac{du}{ds} = c(x(s), y(s), u(s))$$
$\Rightarrow$ yields the solution as a fct of (r, s), i.e. a parametric soln. $u(r, s)$

(4) If req'd invert $x(r,s)$ & $y(r,s)$ to $s(x,y)$ & $r(x,y)$

Then

$$u(r,s) = u(r(x,y), s(x,y)) = u(x,y)$$

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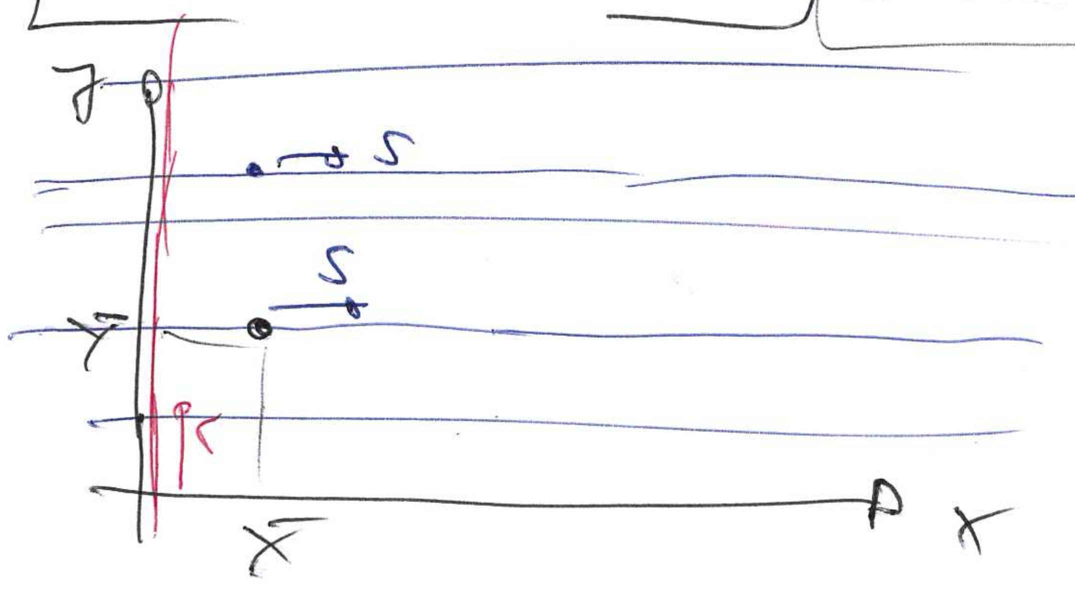
Example: $\frac{\partial u}{\partial x} = 0$ for $u(x,y)$

Integrate: $u(x,y) = f(y)$

(1) Char. eqns:

$$\boxed{\begin{aligned} \frac{dx}{ds} &= a(x,y) = 1 \\ \frac{dy}{ds} &= b(x,y) = 0 \end{aligned}} \quad \boxed{\begin{aligned} x(s) &= s + \bar{x} \\ y(s) &= \bar{y} \end{aligned}}$$



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$\Rightarrow$ characteristics are horizontal lines in (x, y) plane.

Along any of those lines

$$\left[\frac{du}{ds} = c(x, y, u) = 0\right] \Rightarrow \boxed{u(s) = U}$$

② Prescribe form along some initial line. E.g. impose u along the y axis:

$$\begin{pmatrix} X(r) \\ Y(r) \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix} \quad ; \text{ es. } U(r) = r^2$$

$$\left[\underline{\text{i.e.}} \quad u(x=0, y) = y^2 \right]$$

The characteristics become

$$\boxed{\begin{aligned} x(r, s) &= s + X(r) = s \\ y(r, s) &= Y(r) = r \end{aligned}}$$

Into form:

$$u(s) = U$$

$$\boxed{u(r, s) = U(r) = r^2}$$

③ Explicit form: invert $x(r,s)$ & $y(r,s)$ 7

$s = x$

$r = y$

$u = r^2 = y^2$

$$\boxed{u(x,y) = y^2}$$