LECTURE 7

1

Write PDE in operator form

$$\mathcal{L}u = f$$

f only fct of (x,y)

$\mathcal{L}$ is linear if

~~$\mathcal{L}(\ldots)$~~ $\mathcal{L}(u+cv) = \mathcal{L}(u) + c\mathcal{L}(v)$

c const.

Examples

(a) $$2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 7u$$

$$\mathcal{L}u = 0$$

$$\mathcal{L}u = 2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} - 7Iu$$

I identity operator.

(b) $$x\frac{\partial u}{\partial x} + x^2 y\frac{\partial u}{\partial y} = 3x$$

$$\mathcal{L}u = 3x$$

$$\mathcal{L} = x\frac{\partial}{\partial x} + x^2 y\frac{\partial}{\partial y}$$

Linear ?

(e) $f(u) = 2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} - 7u$

Test:

$$f(\underbrace{u}_{u=u(x,y)} + \underbrace{c}_{\text{constant}}\underbrace{v}_{v(x,y)}) = 2\frac{\partial}{\partial x}(u+cv) + 3\frac{\partial}{\partial y}(u+cv) - 7(u+cv)$$

$$= 2\frac{\partial u}{\partial x} + c\,2\frac{\partial v}{\partial x} + 3\frac{\partial u}{\partial y} + c\,3\frac{\partial v}{\partial y} - 7u - c\,7v$$

$$= 2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} - 7u + c\left(2\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} - 7v\right)$$

$$= f(u) + c\,f(v)$$

$\Rightarrow$ linear.

(e) $f(u) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - u^2$

L3

Linear? Replace u by $u+cv$

$$I(u+cv) = \left(\frac{\partial(u+cv)}{\partial x}\right)^2 + \left(\frac{\partial(u+cv)}{\partial y}\right)^2 - (u+cv)^2$$

$$= \left(\frac{\partial u}{\partial x} + c\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + c\frac{\partial v}{\partial y}\right)^2 - (u+cv)^2$$

$$= \underline{\left(\frac{\partial u}{\partial x}\right)^2} + 2\frac{\partial u}{\partial x} c \frac{\partial v}{\partial x} + \underline{c^2\left(\frac{\partial v}{\partial x}\right)^2}$$

$$+ \quad \cdots$$

$$- \left(\underline{u^2} + 2cvu + \underline{c^2v^2}\right)$$

$$= Iu + c^2 Iv + 2c\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} - uv\right)$$

$$\neq Iu + c\,Iv \implies \text{nonlinear.}$$

A few more definitions:

Below: $u = u(x,y)$

Definition

A linear first order PDE can be written in the form

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} =$$

$$= c(x,y)\,u + d(x,y)$$

In operator notation:

$$\mathcal{L}u = d(x,y)$$

$$\mathcal{L} = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} - c(x,y)\,\underline{I}$$

Def: A semi-linear 1st order PDE can be written as

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y,u)$$

Note c: depends nonlinearly on u!

<u>Def:</u> A quasi-linear 1^{st} order PDE can be written as

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$$

<u>Note:</u> At least one of a, b depend on u.

<u>Note:</u> there is a hierarchy of increasing complexity:

Linear → semi-linear → quasilinear → fully nonlinear.

These classifications are mutually exclusive.

Sometimes some algebra is needed to classify the PDE correctly:

<u>E.g.</u>

$$\frac{y}{u}\frac{\partial u}{\partial x} + \frac{x}{u}\frac{\partial u}{\partial y} = xy$$

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} - xyu = 0$$

is actually linear

(b) The method of characteristics

This method is applicable to quasilinear PDEs

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$$

but we will derive the method for semi-linear eqns.

$$\boxed{a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y,u)}$$

How do we "solve" this?
want $u(x,y)$

Recall the directional derivative: