LECTURE 6 (Wed 10/10)

# §2 First order partial diff. eqns (PDE) (1

2.1. Def: A $1^{st}$ order PDE is an eqn. of the form:

$$F(x_1, x_2, \ldots x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n}) = 0$$

for the unknown fct

$$u = u(x_1, x_2, \ldots, x_n).$$

Note: Highest derivative of $u$ w.r.t. any of the $x_i$ is of first order.

Examples $n = 2$; indep. vars: $(x_1, x_2) = (x, y)$

(a) $2 \frac{\partial u}{\partial x} + 3 \frac{\partial u}{\partial y} = 7u$ ~~H~~ HODI

(b) $x \frac{\partial u}{\partial x} + x^2 y \frac{\partial u}{\partial y} = 3x$ IH

(c) $x\frac{\partial u}{\partial x}+u^2x\frac{\partial u}{\partial y}=3x$ IH (2)

(d) $\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}=u$ ~~...~~ H NHFOD1

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?.?. Def: A PDE is called homogeneous (in u) if every term contains u & is called inhomog. if it contains at least one term indep. of u.

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Given any PDE, we may collect all terms involving u on the LHS & all other terms on the RHS. This turns the PDE into

$$\mathcal{L}u=f$$

$\mathcal{L}$ is a differential operator

f is indep of u.

## 2.2. Definition:

(i) A fct $F(u)$ is said to be homogeneous of degree $m$ (in $u$) if for all $\lambda \in \mathbb{R}$

$$F(\lambda u) = \lambda^m F(u)$$

(ii) A PDE of the form $F(u) = 0$ is called homog. of degree $m$ if $F$ is homog. of degree $m$. ~~[otherwise it is called inhomogeneous]~~

**Examplo:**

$$F(u) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - u^2 = 0$$

is homog. of degree 2.

**Proof:** Replace $u$ by $\lambda u$

$$F(\lambda u) = \left(\frac{\partial(\lambda u)}{\partial x}\right)^2 + \left(\frac{\partial(\lambda u)}{\partial y}\right)^2 - (\lambda u)^2$$

$$= \lambda^2 \underbrace{\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - u^2\right]}_{F(u)}$$

Example:

(f) The PDE

$$F(u) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - u = 0$$

is inhomogeneous*, because

$$F(\lambda u) = \lambda^2\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) - \lambda u = 0$$

## 2.3. Definition

If we write the PDE as

$$\mathcal{L}(u) = f \quad \text{(as above)}$$

then $\mathcal{L}(u)$ is called linear if for any two fcts $u$ and $v$ and any constant $c$

(i) $\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v)$

(ii) $\mathcal{L}(cu) = c\,\mathcal{L}(u)$

otherwise it is called non linear.

Notes: (i) Criterion (ii) implies that $\mathcal{L}$ is homog. of degree 1.

(ii) To see if a PDE is linear it suffices to substitute $cu+v$ for $u$. The PDE is linear iff

$$\mathcal{L}(cu+v) = c\,\mathcal{L}(u) + \mathcal{L}(v).$$