Alternative way to do surface integrals

$$\iint \phi(x,y,z)\, dS \quad \text{over}$$

some surface.

Parametrise surface by 2 coords:

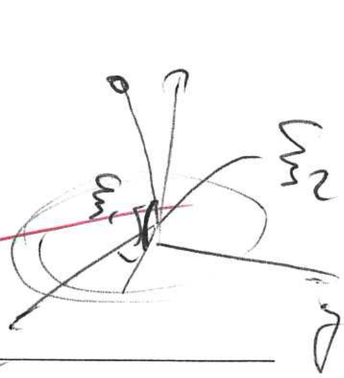
$$\underline{r} = \underline{r}(\xi_1, \xi_2)$$

e.g. <u>cylinder</u>

$$\underline{r} = \begin{pmatrix} a\cos\xi_1 \\ a\sin\xi_1 \\ \xi_2 \end{pmatrix} \qquad \xi_1 \in (0, 2\pi) \quad \xi_2 \in (0, L)$$

<u>sphere</u>

$$\cancel{\underline{r} = \begin{pmatrix} a\cos\xi_1 \sin\xi_2 \\ a\sin\xi_1 \sin\xi_2 \\ a\cos\xi_2 \end{pmatrix}}$$



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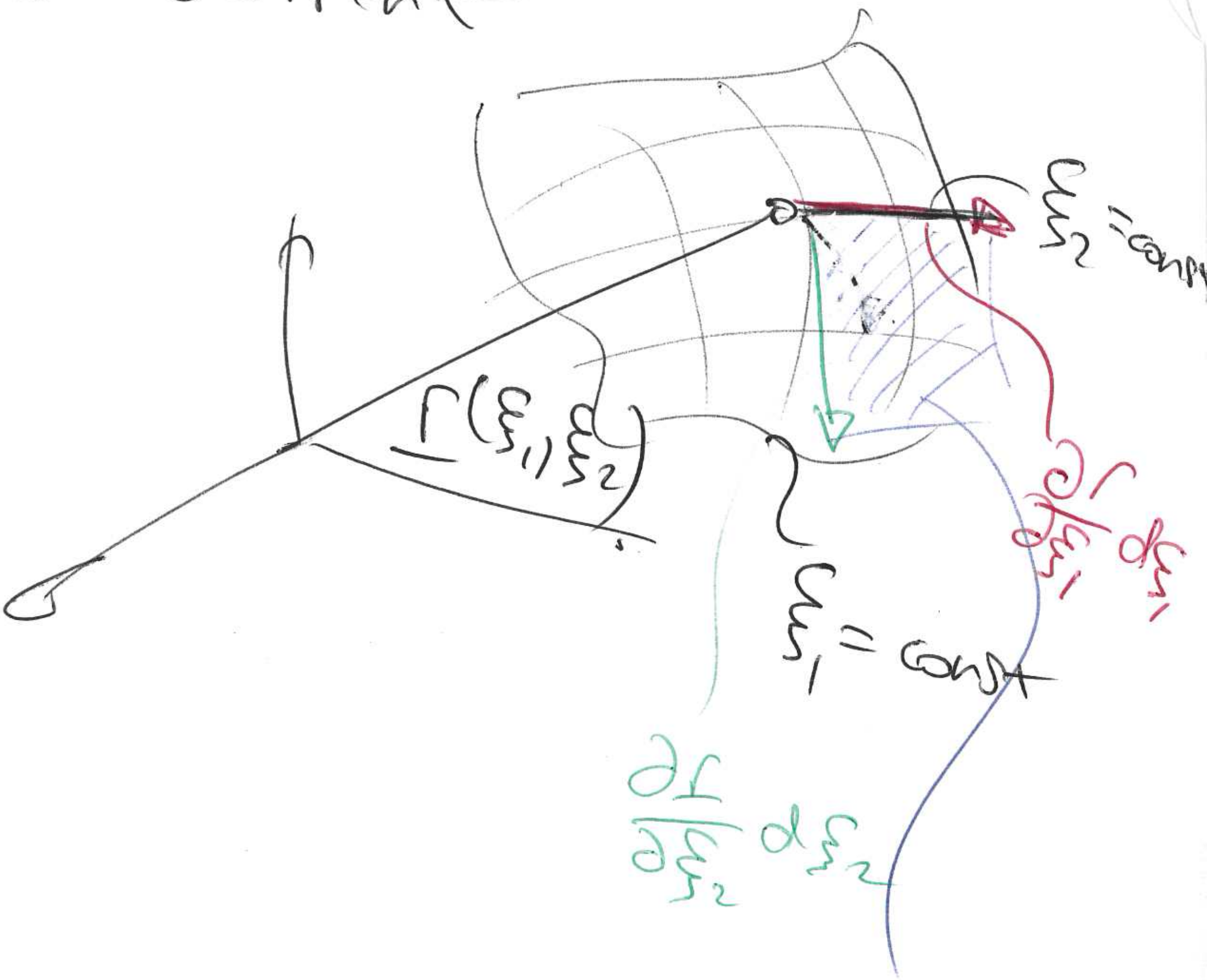
Area element

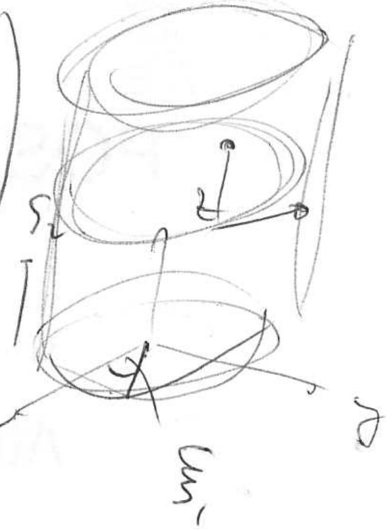


$$dS = \left| \frac{\partial \underline{r}}{\partial \xi_1} \times \frac{\partial \underline{r}}{\partial \xi_2} \right| d\xi_1 \, d\xi_2$$

$$\phi = x^2 + y^2 + z^2$$

over cylinder of radius a & length L.

$$\underline{r}(\xi_1, \xi_2) = \begin{pmatrix} a\cos\xi_1 \\ a\sin\xi_1 \\ \xi_2 \end{pmatrix}$$



$$\xi_1 \in (0, 2\pi]$$

$$\xi_2 \in (0, L]$$

$$\frac{\partial r}{\partial \xi_1} = \begin{pmatrix} -a\sin\xi_1 \\ a\cos\xi_1 \\ 0 \end{pmatrix} \qquad \frac{\partial r}{\partial \xi_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$dS = \underbrace{\left| \frac{\partial r}{\partial \xi_1} \times \frac{\partial r}{\partial \xi_2} \right|}_{} d\xi_1 \, d\xi_2$$

$$\underbrace{\left| \begin{pmatrix} a\cos\xi_1 \\ a\sin\xi_1 \\ 0 \end{pmatrix} \right|}_{} d\xi_1 \, d\xi_2$$

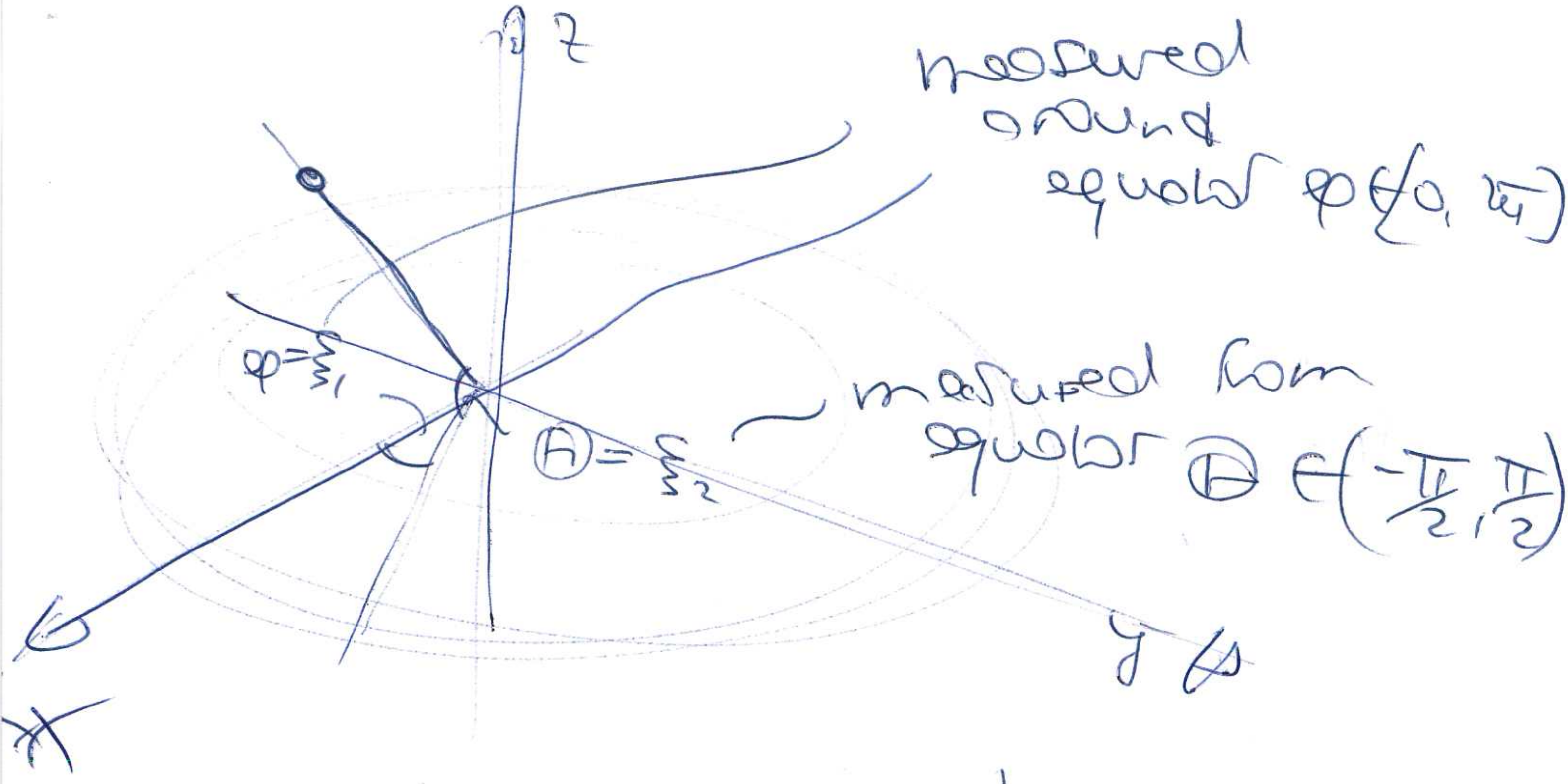
$$dS = a \, d\xi_1 \, d\xi_2$$

$$I = \iint \phi \, dS =$$

$$\int_{\xi_2=0}^{L} \int_{\xi_1=0}^{2\pi} \Big(\underbrace{a^2 \cos^2 \xi_1 + a^2 \sin^2 \xi}_{a^2} + \xi_2^2 \Big) a \, d\xi_1 \, d\xi_2$$

$$\int_{\xi_2=0}^{L} \int_{\xi_1=0}^{2\pi} \left(a^2 + \xi_2^2 \right) a \, d\xi_1 \, d\xi_2$$

$$I = 2\pi \left(a^3 L + a \frac{L^3}{3} \right)$$



$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \cos \xi_2 \cos \xi_1 \\ a \cos \xi_2 \sin \xi_1 \\ a \sin \xi_2 \end{pmatrix}$$

$$\frac{\partial \vec{r}}{\partial \xi_1} = a \begin{pmatrix} -\cos \xi_2 \sin \xi_1 \\ \cos \xi_2 \cos \xi_1 \\ 0 \end{pmatrix}$$

$$\frac{\partial \vec{r}}{\partial \xi_2} = a \begin{pmatrix} -\sin \xi_2 \cos \xi_1 \\ -\sin \xi_2 \sin \xi_1 \\ \cos \xi_2 \end{pmatrix}$$

$$\left|\frac{\partial r}{\partial \xi_1} \times \frac{\partial r}{\partial \xi_2}\right| = \left| a^2 \begin{pmatrix} -\cos\xi_2 \sin\xi_1 \\ \cos\xi_2 \cos\xi_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -\sin\xi_2 \cos\xi_1 \\ -\sin\xi_2 \sin\xi_1 \\ \cos\xi_2 \end{pmatrix} \right|$$

$$= a^2 \cos\xi_2 \left| \begin{pmatrix} \cos\xi_1 \cos\xi_2 - 0 \\ 0 + \cos\xi_2 \sin\xi_1 \\ \sin^2\xi_1 \sin\xi_2 + \cos^2\xi_1 \sin\xi_2 \end{pmatrix} \right|$$

$$= a^2 \cos\xi_2 \left| \begin{pmatrix} \cos\xi_1 \cos\xi_2 \\ \sin\xi_1 \cos\xi_2 \\ \sin\xi_2 \end{pmatrix} \right|$$

$$= a^2 \cos\xi_1 \sqrt{\cos^2\xi_2 \left(\cos^2\xi_1 + \sin^2\xi_1\right) + \sin^2\xi_2}$$

$$= a^2 \cos\xi_1$$

$$\underline{dS = a^2 \cos\xi_1 \, d\xi_1 \, d\xi_2}$$