(LECTURE 20)

Stokes theorem

$$\oint \underline{F} \cdot d\underline{r} = \iint (\nabla \times \underline{F}) \cdot \underline{n} \, dS$$

boundary of open surface

open surface

Example:

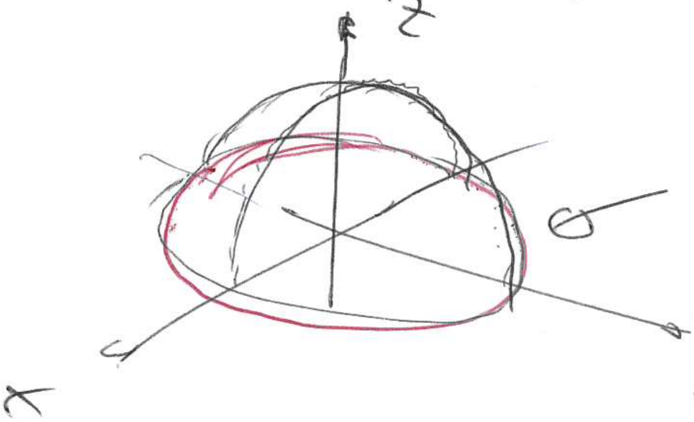
$$\underline{F} = y\,\underline{i} - x\,\underline{j}$$

$$= F_x\,\underline{i} + F_y\,\underline{j} + F_z\,\underline{k}$$

$$F_x = y \; ; \; F_y = -x \; ; \; F_z = 0$$

Apply Stokes thm for hemisphere

$$x^2 + y^2 + z^2 = 9 \; ; \; z \geq 0$$



σ — boundary: circle of radius 3.

LHS: $$\oint \underline{F} \cdot d\underline{r} = \oint \underline{F} \cdot \frac{d\underline{r}}{dt} dt$$

where $\underline{r}(t)$ parametrises boundary.

Set $\varphi = t$

$$\underline{r}(\varphi) = \begin{pmatrix} 3\sin\varphi \\ 3\cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} x(\varphi) \\ y(\varphi) \\ z(\varphi) \end{pmatrix}$$

$$\varphi : 0 \to 2\pi$$

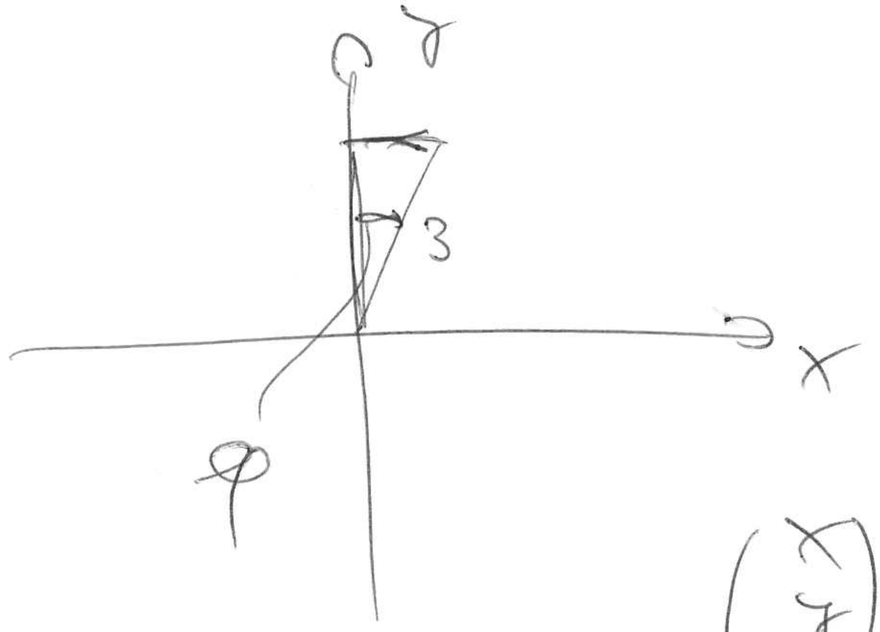
$$\frac{dr}{d\varphi} = \begin{pmatrix} 3\cos\varphi \\ -3\sin\varphi \\ 0 \end{pmatrix}$$

$$\underline{F}(x,y,z) = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} = \begin{pmatrix} 3\cos\varphi \\ -3\sin\varphi \\ 0 \end{pmatrix}$$

$$\int_0^{2\pi} \underbrace{\begin{pmatrix} 3\cos\varphi \\ -3\sin\varphi \\ 0 \end{pmatrix}}_{\underline{F}(\varphi)} \cdot \cancel{\begin{matrix} 3\sin\varphi \\ 3\cos \end{matrix}} \underbrace{\begin{pmatrix} 3\cos\varphi \\ -3\sin\varphi \\ 0 \end{pmatrix}}_{\frac{dr}{d\varphi}} d\varphi$$

$$\int_0^{2\pi} \underbrace{\left(9\cos^2\varphi + 9\sin^2\varphi\right)}_{9} d\varphi$$

$$= 18\pi$$

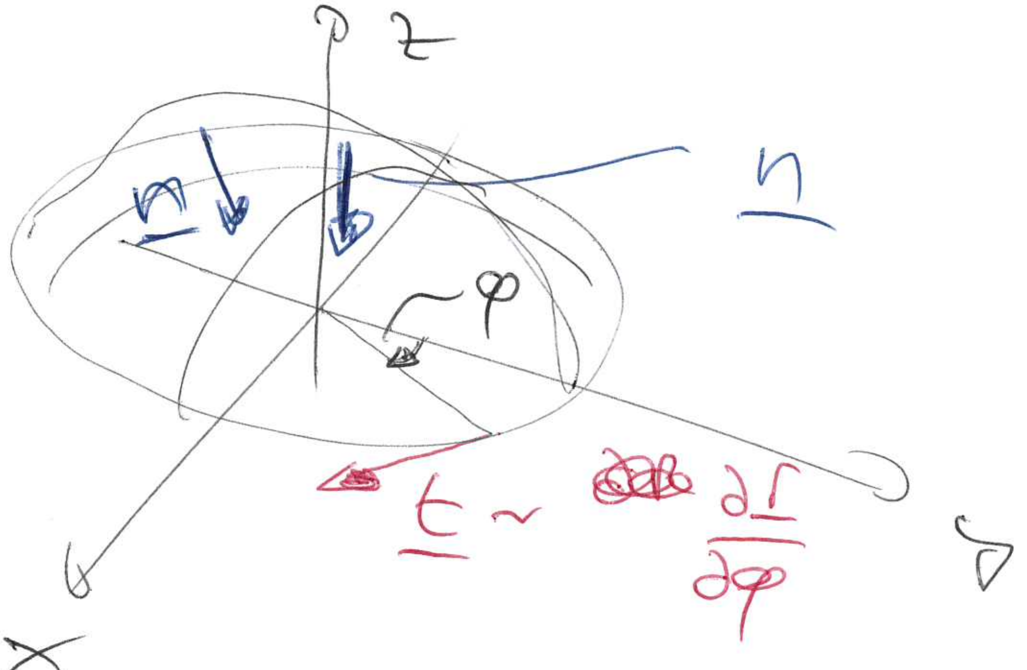


$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3\sin\varphi \\ 3\cos\varphi \\ 0 \end{pmatrix}$$

Note: φ increases in clockwise direction!

RHS: $\iint (\nabla \times \underline{F}) \cdot \underline{n} \, dS$

Such that $\underline{n}$ follows from the right-hand screw rule.



$$\nabla \times \underline{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} = \begin{pmatrix} 0-0 \\ 0-0 \\ -1-1 \end{pmatrix}$$

$$= -2\,\underline{k}$$

What is $\underline{n}$? To satisfy the right-hand screw rule $\underline{n}$ must be the inner unit normal to the sphere!
In previous question:

$$\underline{n}_{outer} = \frac{x}{a}\,\underline{i} + \frac{y}{a}\,\underline{j} + \frac{z}{a}\,\underline{k}$$

is outer unit normal on a sphere of radius a.

$$\underline{n} = -\frac{x}{a}\underline{i} - \frac{y}{a}\underline{j} - \frac{z}{a}\underline{k}$$

$$\oiint \underbrace{(\nabla \times \underline{F})} \cdot \underline{n}\, dS$$

$$\oiint \left(-2\underline{k} \cdot \left(-\frac{x}{a}\underline{i} - \frac{y}{a}\underline{j} - \frac{z}{\underset{\hookrightarrow 3}{a}}\underline{k}\right)\right) dS$$

$$\oiint +\frac{2}{3} z \, dS$$

$$= \frac{2}{3} \oiint z \, dS$$

This is a surface integral of the form

$$\iint \underbrace{\phi(x,y,z)}_{z} \, dS$$

Recall: use explicit representation of upper hemisphere, projected into x-y plane.

$$z = f(x,y) = \sqrt{9 - x^2 - y^2}$$

$$\frac{2}{3}\iint \underbrace{\phi(x,y,f(x,y))}_{\sqrt{9-x^2-y^2}} \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy$$

$$f(x,y) = (9-x^2-y^2)^{1/2}$$

$$\frac{\partial f}{\partial x} = \frac{\cancel{4}\ -\cancel{2}x}{\cancel{2}\sqrt{9-x^2-y^2}} = \frac{-x}{\sqrt{9-x^2-y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{9-x^2-y^2}}$$

$$I = \frac{2}{3}\iint \sqrt{9-x^2-y^2}\sqrt{1+\frac{x^2}{9-x^2-y^2}+\frac{y^2}{9-x^2-y^2}}\,dx\,dy$$

$$= \frac{2}{3}\iint \cancel{\sqrt{\ }}\sqrt{\frac{9-\cancel{x^2}-\cancel{y^2}+\cancel{x^2}+\cancel{y^2}}{\cancel{9-x^2-y^2}}}\,dx\,dy$$

$$= \frac{2}{\cancel{3}}\iint \cancel{3}\,dx\,dy$$

$$= 2\iint dx\,dy$$

integral over the project. of the surface into x-y plane.

⇒ area of the circle of radius 3

9π

$I_{RHS} = 18\pi$ again!