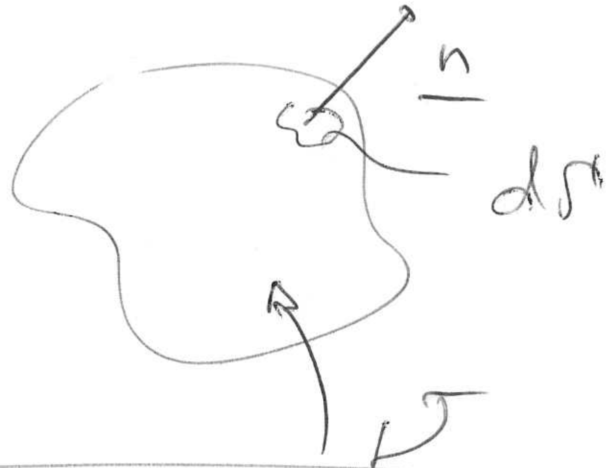
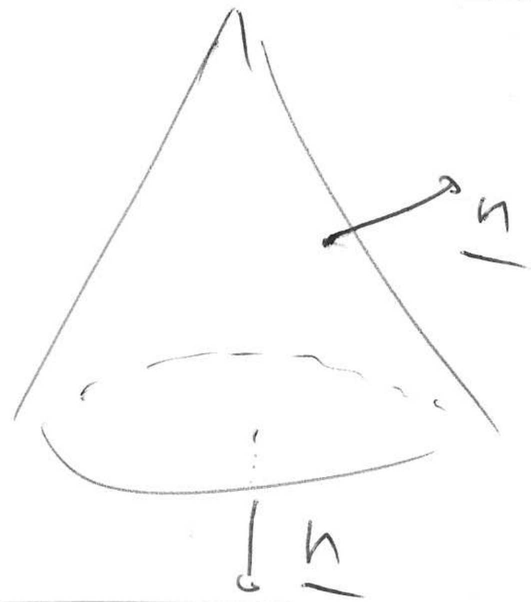
Gauss / divergence theorem 19

$$\iint_S \underline{F} \cdot \underline{n} \, dS = \iiint_V \nabla \cdot \underline{F} \, dV$$





Total flux of $\underline{F}$ over the closed surface S = integral of div $\underline{F}$ over enclosed volume V.

Stokes' theorem

Let C denote a piecewise smooth simple closed curve in space. Let $\underline{t}$ denote the unit tangent to C at $\underline{r}$ then $d\underline{r} = \underline{t}\, ds$ where s is the arclength along C.

$\underline{r}(s)$.

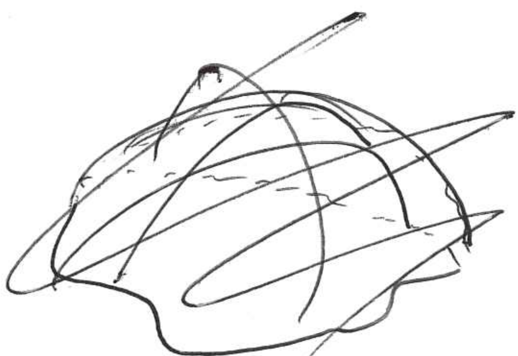
<u>Def</u>: The tangential line integral on C is

$$\int_C \underline{F} \cdot d\underline{r} = \int \underline{F} \cdot \underbrace{\frac{d\underline{r}}{ds}}_{\underline{t}}\, ds = \int \underline{F}\, \underline{t}\, ds$$

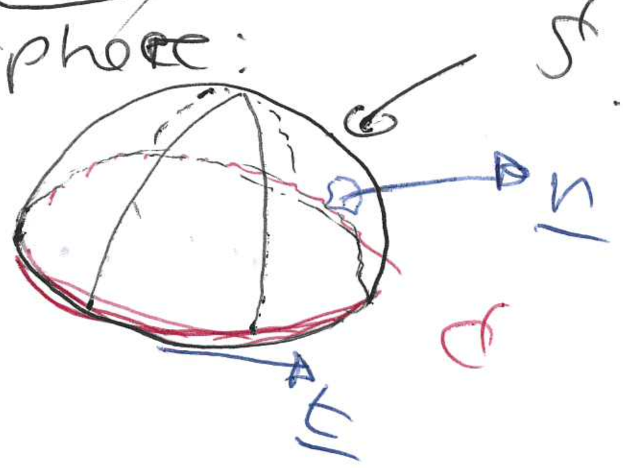
For a closed C this is called the <u>circulation</u>.

Theorem:

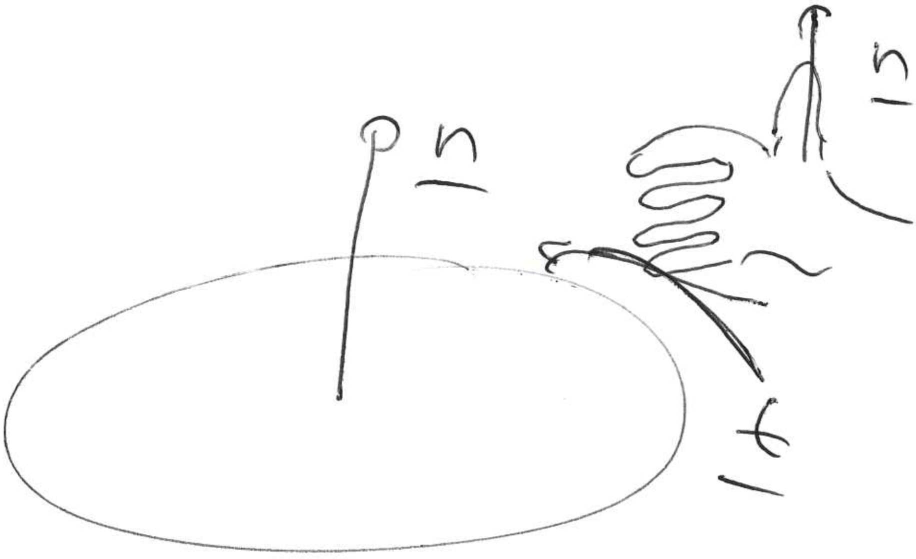
- Let S be a piecewise smooth orientable open surface in space with boundary C.
- $\underline{n}$ is the pos. unit normal on that surface
- $\underline{t}$ is unit tangent to C.
- $\underline{F}$ is a vector field: continuous & has continuous ~~de~~ 1st derivs.



E.g. hemisphere:



$\underline{t}$ & $\underline{n}$ such that the direction of $\underline{t}$ is obtained from the direction of $\underline{n}$ via the right-hand screw rule.



Then:

$$\boxed{\oint \underline{F}\cdot d\underline{r} = \oint \underline{F}\cdot\frac{d\underline{r}}{ds}\, ds = \iint_{S'} (\nabla\times\underline{F})\cdot\underline{n}\, dS'}$$

Tangential line integral along C' (circulation) = normal surface integral of $\nabla\times\underline{F}$ over S'.

Example: verify Gauss thm for:

$$\underline{F} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}$$

& a sphere of radius a, centered at origin.

$$\iint \underline{F}\cdot\underline{n}\,dS = \underbrace{\iiint \nabla\cdot\underline{F}\,dV}$$

RHS: $\nabla\cdot\underline{F} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} =$

$$= 1 + 1 + 1 = \underline{\underline{3}}$$

$$\iiint \nabla\cdot\underline{F}\,dV = 3\underbrace{\iiint dV}_{\text{volume of sphere}} = 3\cdot\frac{4}{3}\pi a^3$$

$$\underline{\underline{= 4\pi a^3}}$$

LHS: $\iint \underline{F} \cdot \underline{n} \, dS$

Surface: defined by

$$F(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0$$

$$\underline{n} = \frac{\nabla F}{|\nabla F|}$$

$$\nabla F = 2x\,\underline{i} + 2y\,\underline{j} + 2z\,\underline{k}$$

$$|\nabla F| = \sqrt{(2x)^2 + (2y)^2 + (2z)^2}$$

$$= 2\sqrt{x^2 + y^2 + z^2} = 2a$$

because we're on surface.

$$\underline{n} = \frac{x}{a}\underline{i} + \frac{y}{a}\underline{j} + \frac{z}{a}\underline{k}$$

$$\iint \underline{F} \cdot \underline{n} \, dS =$$

$$= \iint (x\underline{i} + y\underline{j} + z\underline{k})\left(\frac{x}{a}\underline{i} + \frac{y}{a}\underline{j} + \frac{z}{a}\underline{k}\right) dS$$

$$= \iint \frac{\overbrace{x^2+y^2+z^2}^{a^2}}{a} \, dS$$

$$= \iint a \, dS = a \underbrace{\iint dS}_{\substack{\text{Surface area} \\ \text{of sphere} \\ = 4\pi a^2}} = \underline{4\pi a^3}$$

as above.