LECTURE 18

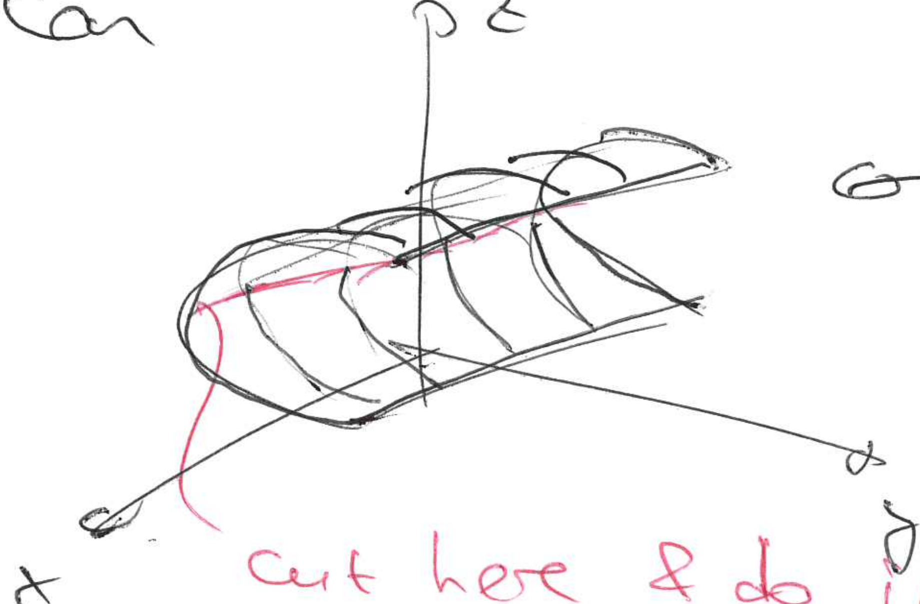
$$I = \iint \phi(x,y,z)\, dS$$

Assume: $S: \; z = f(x,y)$

[Surface can be projected one-to-one into x-y plane!]

$$I = \iint \phi(x,y,f(x,y)) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy$$

Note: If the surface cannot be projected one-to-one into the x-y-plane then break it ~~cut~~ up into components that can



← C-shaped surface.

cut here & do integral over upper & lower half. separately and / or do the equivalent derivation/projection into another coordinate direction.

E.g. S: $x = g(y,z)$

$$I = \iint \phi(g(y,z), y, z) \sqrt{1 + \left(\frac{\partial g}{\partial y}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2} \, dy \, dz$$

etc.

Example:

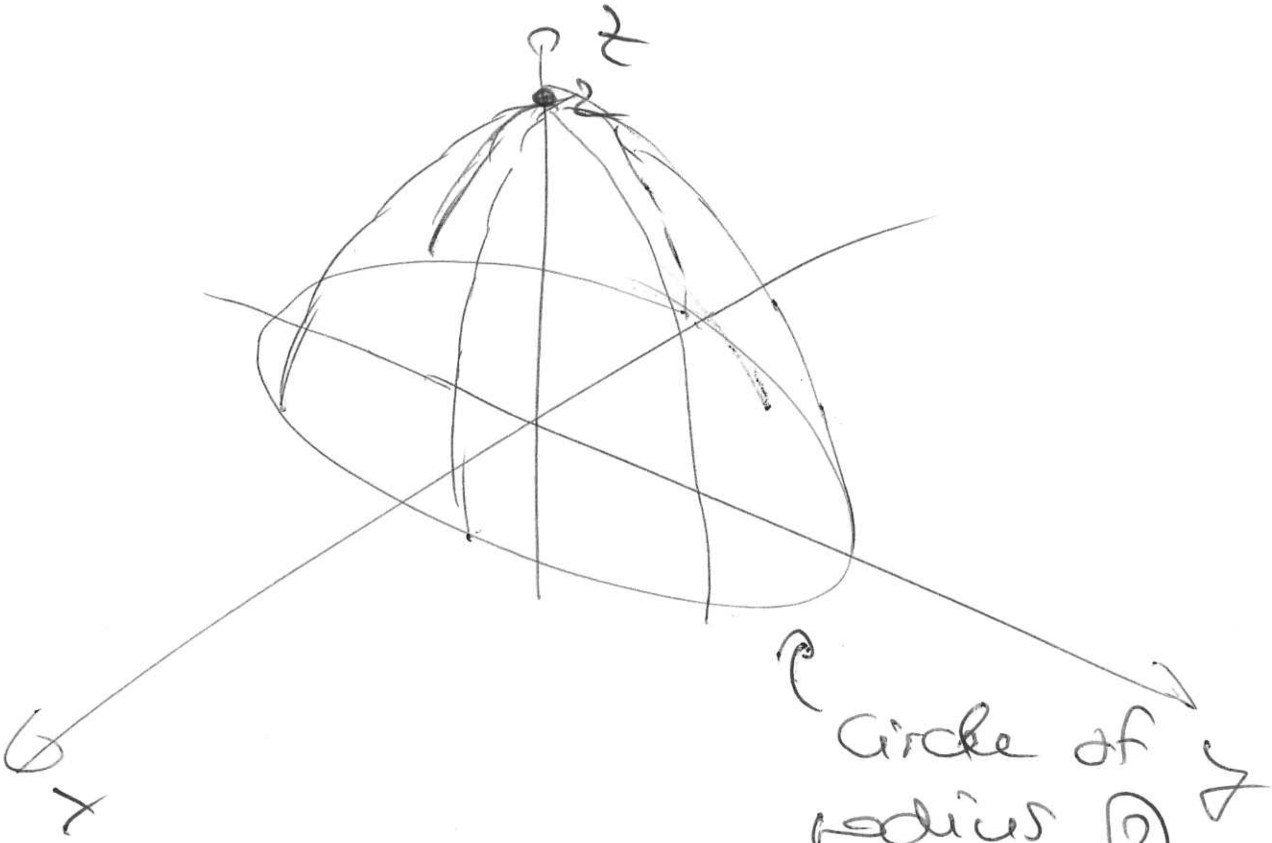
$$I = \iint \phi(x,y,z) \, dS$$

where S is the surface of the paraboloid defined by

$$z = f(x,y) = 2 - x^2 - y^2$$

above the x-y plane.

for $\phi(x,y,z) = 3z$



$$I = \iint \phi(x,y,z)\, \cancel{f} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy$$

$$f(x,y) = 2 - x^2 - y^2$$

$$\frac{\partial f}{\partial x} = -2x; \quad \frac{\partial f}{\partial y} = -2y$$

$$\phi(x,y,f(x,y)) = 3f(x,y) = 3(2-x^2-y^2)$$

$$I = \iint 3(2-x^2-y^2)\sqrt{1+4(x^2+y^2)}\, \underline{dx\, dy}$$

integral over the projection of the surface into x-y plane

$\Rightarrow$ circle with radius $\sqrt{2}$

Switch to polar coords in x-y plane.

$$r^2 = x^2 + y^2$$

$$dx\,dy = r\,dr\,d\varphi$$

Limits: $r: 0 \to \sqrt{2}$

$\varphi: 0 \to 2\pi$

$$I = 3\int_0^{2\pi}\int_0^{\sqrt{2}} (2-r^2)\sqrt{1+4r^2}\, r\,dr\,d\varphi$$

$$I = 6\pi \int_0^{\sqrt{2}} (2-r^2)\sqrt{1+4r^2}\, r\,dr$$

Subst: $u = 1+4r^2$

$$r^2 = \frac{u-1}{4} \;;\; \frac{du}{dr} = 8r$$

$$dr = \frac{1}{8r}du$$

Limits: $r: 0 \to \sqrt{2} \Rightarrow u: 1 \to 9$

$$I = 6\pi \int_1^9 \left(2 - \underbrace{\frac{u-1}{4}}_{r^2}\right) \underbrace{u^{1/2}}_{\sqrt{\ldots}} \underbrace{\cancel{r}\,\frac{1}{8\cancel{r}}\,du}_{dr}$$

$$= \frac{3\pi}{4} \int_1^9 \left(\frac{9}{4} u^{1/2} - \frac{1}{4} u^{3/2}\right) du$$

$$= \frac{3\pi}{4} \left[\frac{9}{4}\frac{2}{3} u^{3/2} \Big|_1^9 - \frac{1}{4}\frac{2}{5} u^{5/2} \Big|_1^9 \right]$$

$$= \frac{111}{10}\pi$$

Divergence & Stokes' theorem

Def:

- A smooth surface has a unique, continuously turning normal at each point. s.d.e

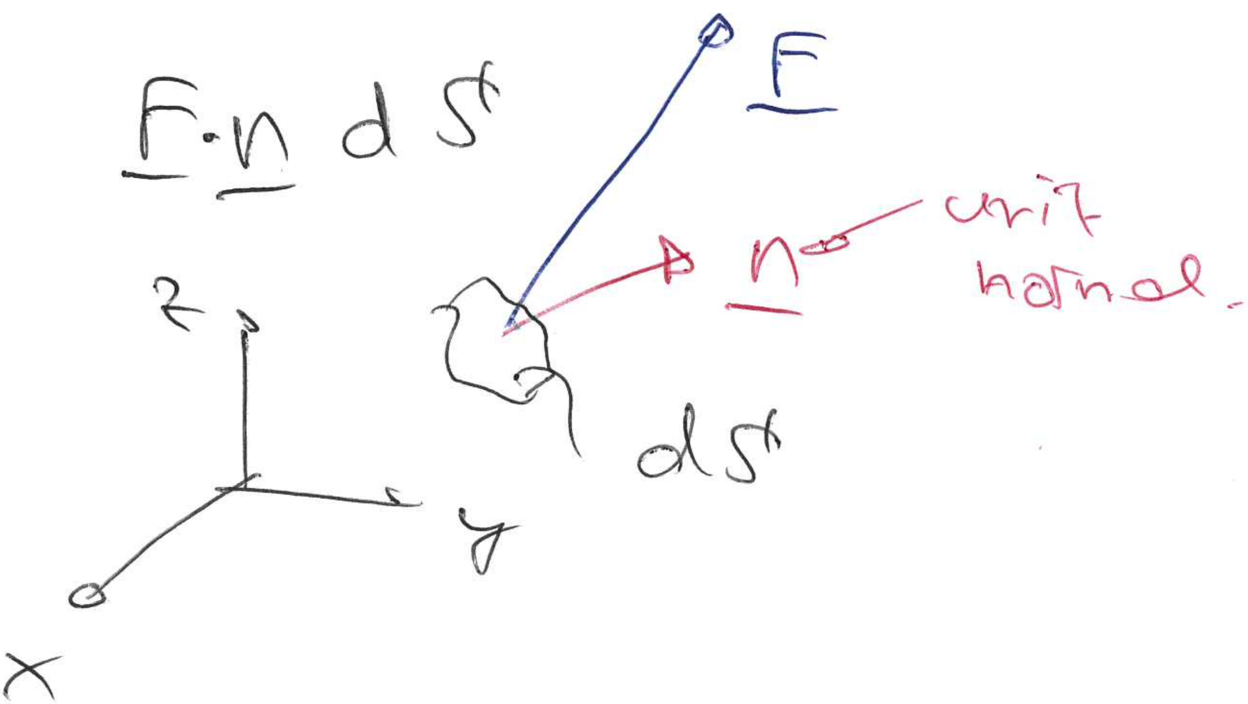
  E.g. sphere.

- A piecewise smooth surface consists of finitely many smooth surfaces (e.g. s.o.t. cube)
- An orientable surface has two sides (counterexample: Möbius surface)
- An open surface has a boundary curve, e.g. hemispherical surface
- A closed surface has no boundary (surface of a sphere)

Flux:

Def: The infinitesimal flux of a vector field $\underline{F}$ across an area element $d\underline{S} = dS\,\underline{n}$ is given by: $\underline{F} \cdot d\underline{S} =$

$\underline{F} \cdot \underline{n} \, dS$



Def: The total flux of $\underline{F}$ over a surface S is

$$\iint \underline{F} \cdot \underline{n} \, dS$$

Theorem 5.2. Divergence theorem

Let V be a closed, bounded region whose boundary S is a piecewise smooth orientable surface.

$\underline{F}$ a vector field defined in V, which is continuous & has continuous 1^{st} derivatives. Let $\underline{n}$ be the outer unit normal on S.

Then:

$$\iint_S \underline{F} \cdot \underline{n} \, dS = \iiint_V \underbrace{\nabla \cdot \underline{F}}_{\text{div } \underline{F}} \, dV$$