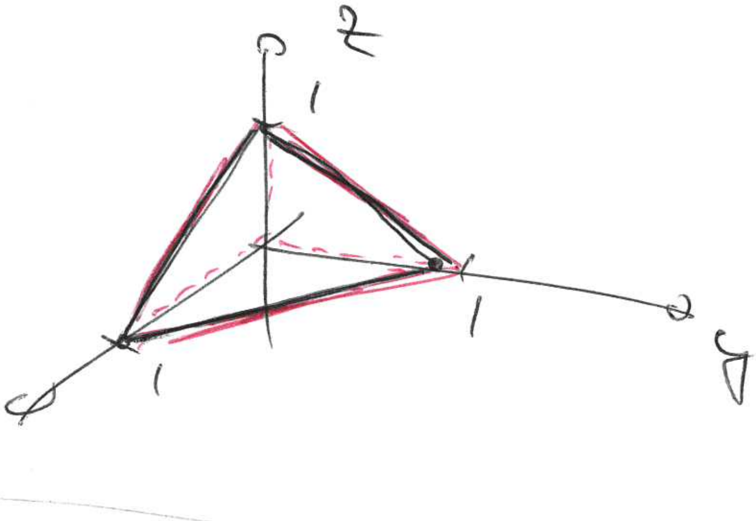
LECTURE 17

Example:

$$I = \iiint_V \underbrace{\frac{1}{(x+y+2z+1)^3}}_{\phi(x,y,z)} \, dV$$

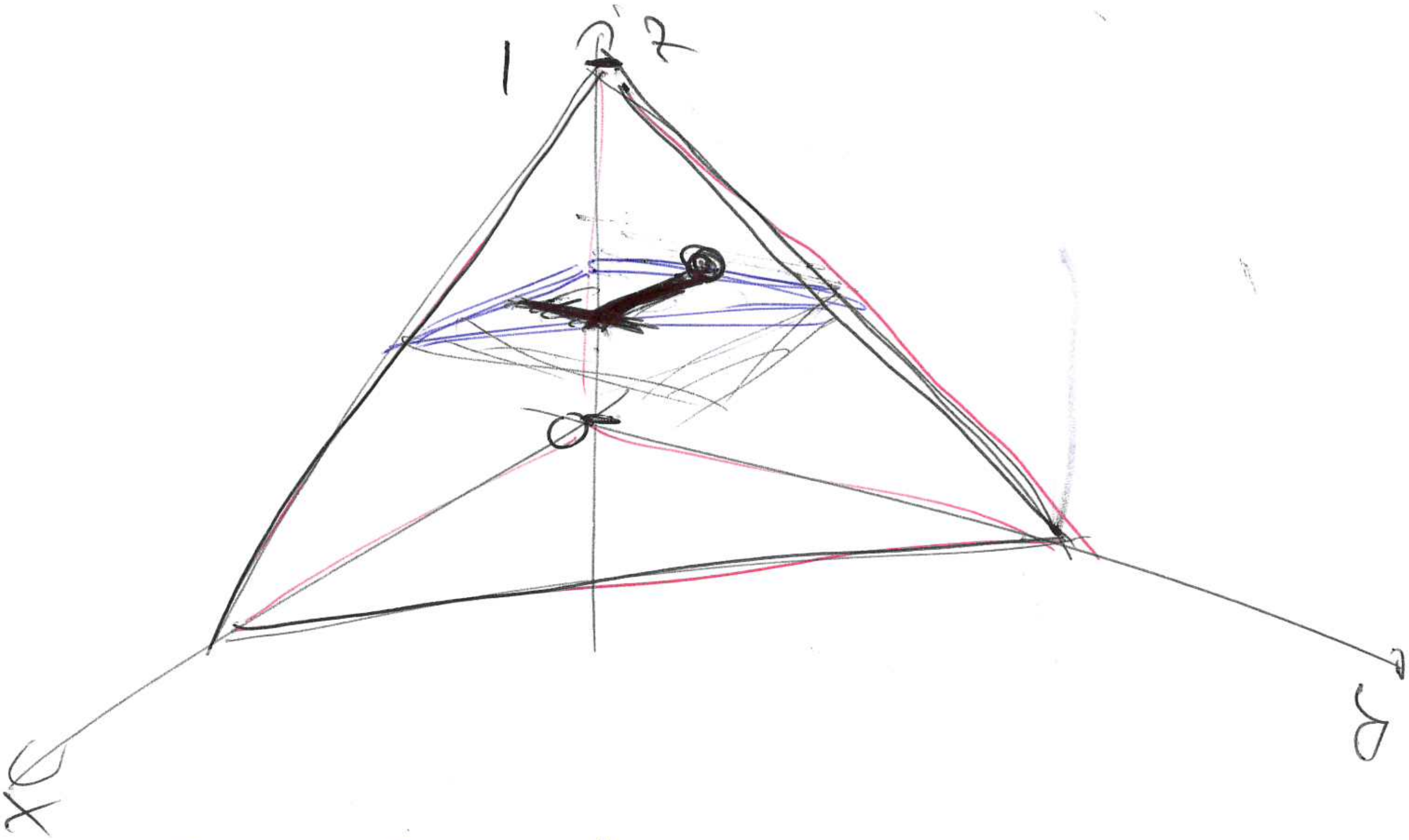
V : tetrahedron enclosed by the coordinate planes & the surface $x+y+z=1$.

[$f(x,y,z) = x+y+z-1 \equiv 0$]



$$I = \int\int\int \phi(x,y,z) \, \underbrace{dx\,dy\,dz}_{dV}$$

Limits?



$z: 0 \to 1$

$y: 0 \to 1 - z$

$x: 0 \to 1 - y - z$

Sanity check

Final x point must be on surface $f(x,y,z) = 0$

clear:

$$F(x,y,z) = x + y + z - 1$$

$$= (1 - y - z) + y + z - 1 = 0$$

✓

$$I = \int_{z=0}^{1} \int_{y=0}^{1-z} \int_{x=0}^{1-y-z} \phi(x,y,z)\, dx\, dy\, dz$$

(see Ex. Sheet 6)

Surface integrals

<u>Recall:</u> Surface in 3D can be defined implicitly as $f(x,y,z) = 0$. or explicitly in the form

$$z = f(x,y)$$

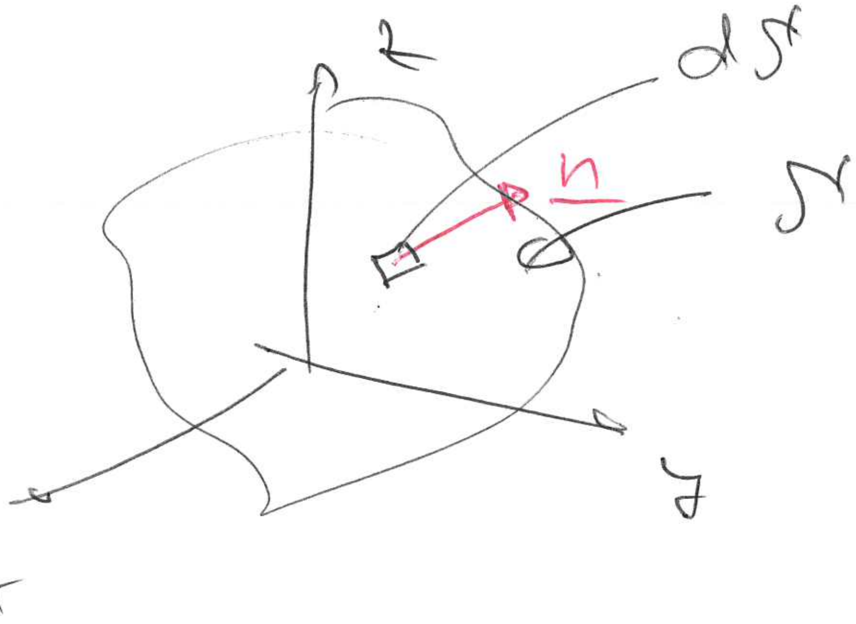
or $y = g(x,z)$ or $x = h(y,z)$

Surface integral:

$$I = \iint_S \underbrace{\phi(x,y,z)}_{\text{evaluated on the surface } S} dS$$

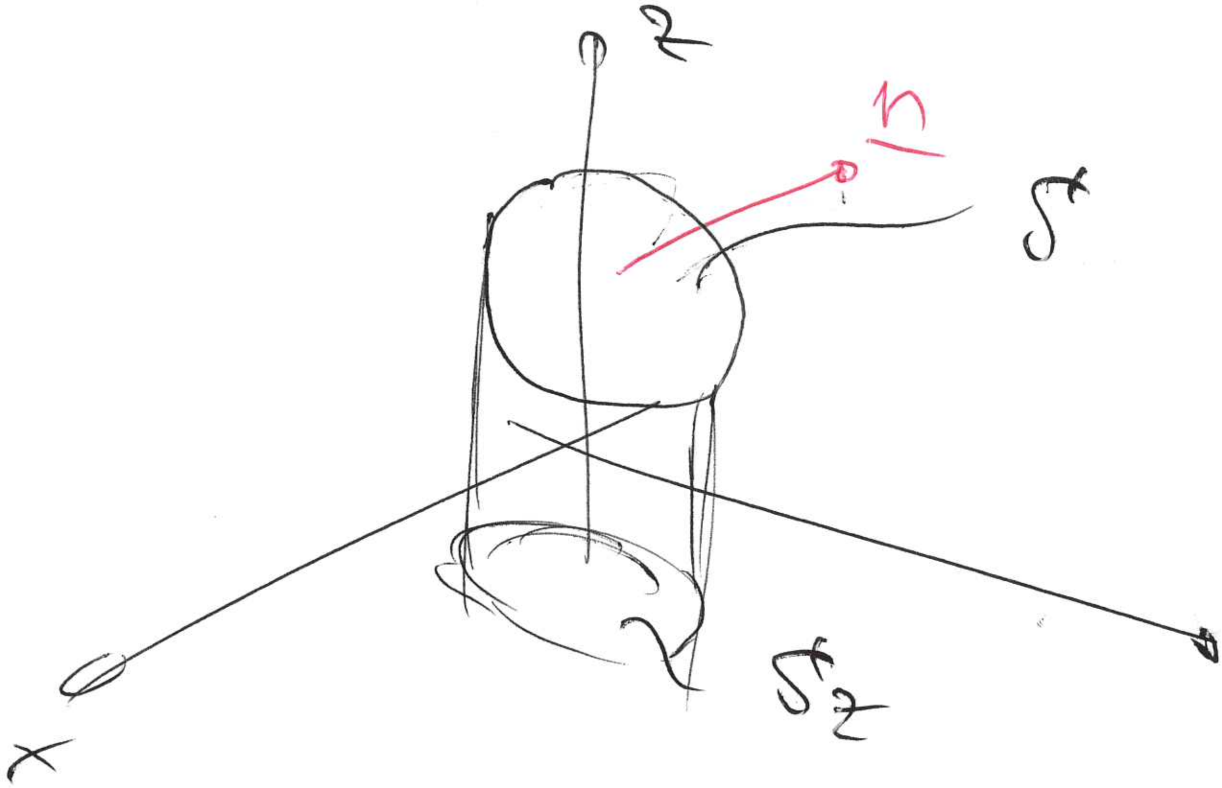
Q: How do we represent the surface S or its constituent elements dS.

A: Define surface element dS by a surface vector:



$$d\underline{S} = \underbrace{dS}_{\text{area of the surface element}} \ \underbrace{\underline{n}}_{\text{unit normal on the surface element.}}$$

works for finite planar area elements too:



$$\underline{S} = S\,\underline{n} \qquad \underline{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$$

Note: $\underline{S}\cdot\underbrace{\underline{k}}_{\substack{\text{unit}\\ \text{vector}\\ \text{in}\\ z\text{-direction}}} = S\,\underbrace{\underline{n}\cdot\underline{k}}_{\substack{\text{cos of}\\ \text{the}\\ \text{angle}\\ \text{between}\\ \underline{n}\ \&\ \underline{k}}} = n_z$

Def:
$$\underline{n}\cdot\underline{i} = n_x = \cos\alpha$$
$$\underline{n}\cdot\underline{j} = n_y = \cos\beta$$
$$\underline{n}\cdot\underline{k} = n_z = \cos\gamma$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are ~~the~~ the directional cosines of that plane.

$$\underline{dS} = \underline{n}\, dS$$

$$d\underline{S}\cdot\underline{k} = \boxed{\underbrace{\underline{n}\cdot\underline{k}}_{n_z}\, dS = dS_z}$$

projection of the scalar area dS into the x-y plane

use this in

$$I = \iint \phi(x,y,z)\, dS$$

$$= \underbrace{\iint \phi(x,y,z) \frac{1}{|\underline{n}\cdot\underline{k}|} dS_z}_{\text{integral in the x-y plane}}$$

Assume: Let S be such that it can be projected one-to-one into the x-y plane, i.e. that you can represent it explicitly as

$$z = f(x, y)$$

What is $\underline{n}$?

Recall:

$$\underline{n} = \frac{\nabla F}{|\nabla F|}$$

where $F(x, y, z) = 0$ on surface.

Choose: $F(x, y, z) = f(x, y) - z = 0$

$$\nabla F = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ -1 \end{pmatrix}$$

$$|\nabla F| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

$\underline{n} \cdot \underline{k} =$

$$\frac{1}{|\nabla F|} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$|\underline{n} \cdot \underline{k}| = \frac{1}{|\nabla F|} = \frac{1}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

$$I = \iint \phi(x, y, z) \frac{1}{|\underline{n} \cdot \underline{k}|} \, dS_2$$

$$\boxed{I = \iint \phi(x, y, f(x, y)) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dx \, dy}$$