LECTURE 16

$$I = \int \underline{F} \cdot d\underline{r}$$

$$\underline{F} = \begin{pmatrix} x^2 y \\ x z \\ -2yz \end{pmatrix} \; ; \; \underline{r}(t) = \begin{pmatrix} 4t \\ 2t^2 \\ t^3 \end{pmatrix}$$

$$t \in (0, 1)$$

$$\int \underline{F} \cdot d\underline{r} = \int \underline{F} \cdot \frac{d\underline{r}}{dt} dt$$

$$\frac{d\underline{r}}{dt} = \begin{pmatrix} 4 \\ 4t \\ 3t^2 \end{pmatrix} \; ; \; \underline{F} = \begin{pmatrix} 32t^4 \\ 4t^4 \\ -4t^5 \end{pmatrix}$$

$$I = \int_{t=0}^{1} \left(128t^4 + 16t^5 - 12t^7\right) dt$$

$$= \frac{803}{30}$$

INTERUPTION (BACK TO
DIV GRAD CURL)

Specific example:

$$u_1 = r \quad u_2 = \varphi \quad u_3 = z$$

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$$

Tangent vectors to coord. lines:

$$\frac{\partial \underline{r}}{\partial u_1} = \frac{\partial \underline{r}}{\partial r} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}; \; h_1 = \left|\frac{\partial \underline{r}}{\partial u_1}\right| = 1 = h_r$$

$$\frac{\partial \underline{r}}{\partial u_2} = \frac{\partial \underline{r}}{\partial \varphi} = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix}; \; h_2 = \left|\frac{\partial \underline{r}}{\partial u_2}\right| = r = h_\varphi$$

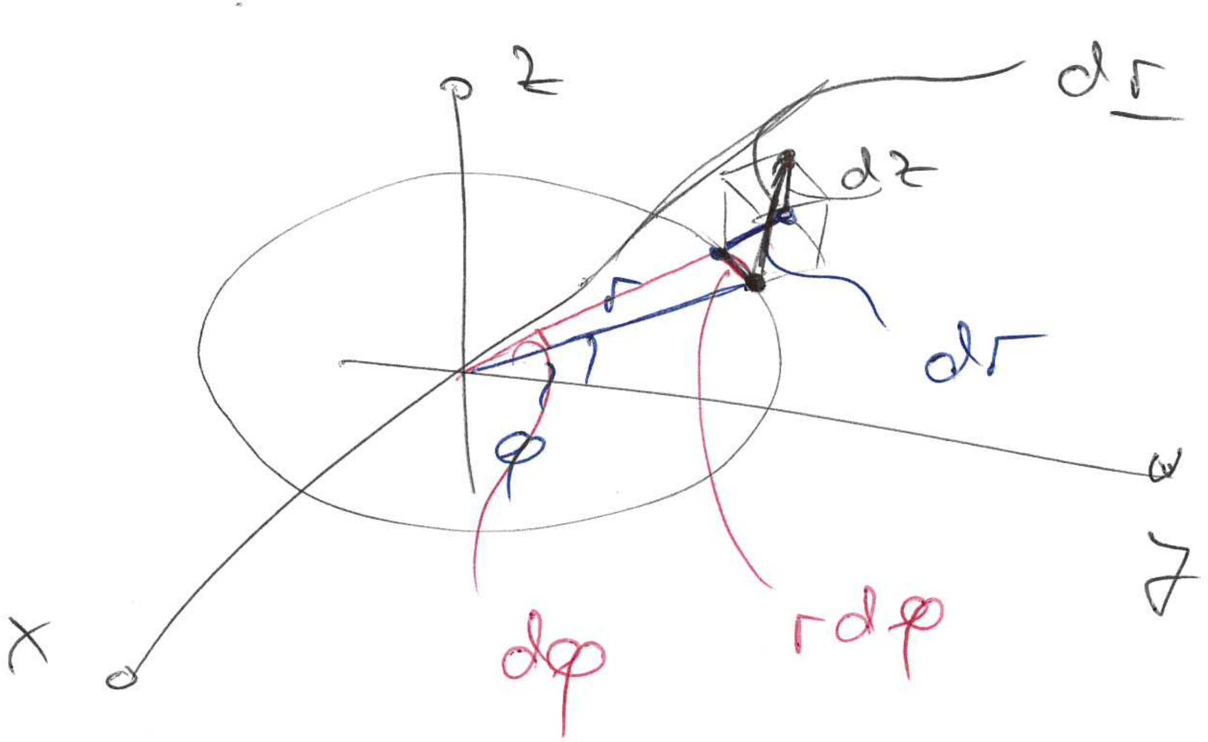
$$\frac{\partial \underline{r}}{\partial u_3} = \frac{\partial \underline{r}}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \; h_z = 1$$

Note: $\frac{\partial \underline{r}}{\partial u_1}$, $\frac{\partial \underline{r}}{\partial u_2}$ & $\frac{\partial \underline{r}}{\partial u_3}$ are orthog.

Arclength:

$$ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

$$ds^2 = dr^2 + r^2 d\varphi^2 + dz^2$$



$|d\underline{r}| = dr$ expression above is correct because of Pythagoras.

Vol. element:

$$dV = \underbrace{h_1 h_2 h_3}_{r} du_1 du_2 du_3$$

$$= r\, dr\, d\varphi\, dz$$

$$= (r\, d\varphi)\, dr\, dz$$

$$dV = J\, dr\, d\varphi\, dz$$

$$J = \left| \frac{\partial(x,y,z)}{\partial(r,\varphi,z)} \right| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$J = \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

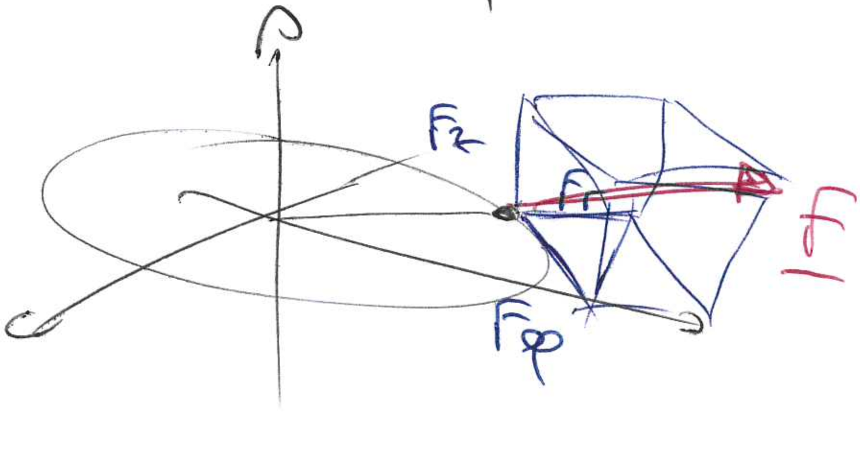
$$= r\cos^2\varphi + r\sin^2\varphi = r \quad \checkmark$$

$$\operatorname{grad}\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1}\underline{e_1} + \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}\underline{e_2} + \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}\underline{e_3}$$

$$\operatorname{grad}\phi = \frac{\partial\phi}{\partial r}\underline{e_r} + \frac{1}{r}\frac{\partial\phi}{\partial\varphi}\underline{e_\varphi} + \frac{\partial\phi}{\partial z}\underline{e_z}$$

~~div~~

$$\underline{F} = F_r\,\underline{e_r} + F_\varphi\,\underline{e_\varphi} + F_z\,\underline{e_z}$$



$$\operatorname{div} \underline{F} = \frac{1}{h_r h_\varphi h_z} \left[\frac{\partial}{\partial r}\left(h_\varphi h_z F_r\right) + \frac{\partial}{\partial \varphi}\left(h_r h_z F_\varphi\right) + \frac{\partial}{\partial z}\left(h_r h_\varphi F_z\right) \right]$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r F_r\right) + \frac{1}{r}\frac{\partial F_\varphi}{\partial \varphi} + \cancel{\frac{1}{r}}\frac{\partial}{\partial z}\left(\cancel{r} F_z\right)$$

$$\operatorname{div} \underline{F} = \frac{1}{r}\frac{\partial}{\partial r}\left(r F_r\right) + \frac{1}{r}\frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial F_z}{\partial z}$$

compare:

$$\operatorname{div} \underline{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

END OF INTERRUPTION.

Volume integrals:

For scalar fields $\phi(x, y, z)$

$$I = \iiint_V \phi(x, y, z)\, dV$$

$$= \iiint \phi(u_1, u_2, u_3)\, \underbrace{h_1 h_2 h_3}_{J}\, du_1\, du_2\, du_3$$

in an orthogonal curvilinear coord system.

For vector fields:

$$\underline{F} = f_x \underbrace{\underline{e_x}}_{\underline{i}} + f_y \underbrace{\underline{e_y}}_{\underline{j}} + f_z \underbrace{\underline{e_z}}_{\underline{k}}$$

$$\iiint \underline{F}\, dV = \underline{e_x} \iiint f_x(x, y, z)\, dV + \underline{e_y} \iiint f_y(x, y, z)\, dV + \underline{e_z} \iiint f_z(x, y, z)\, dV.$$