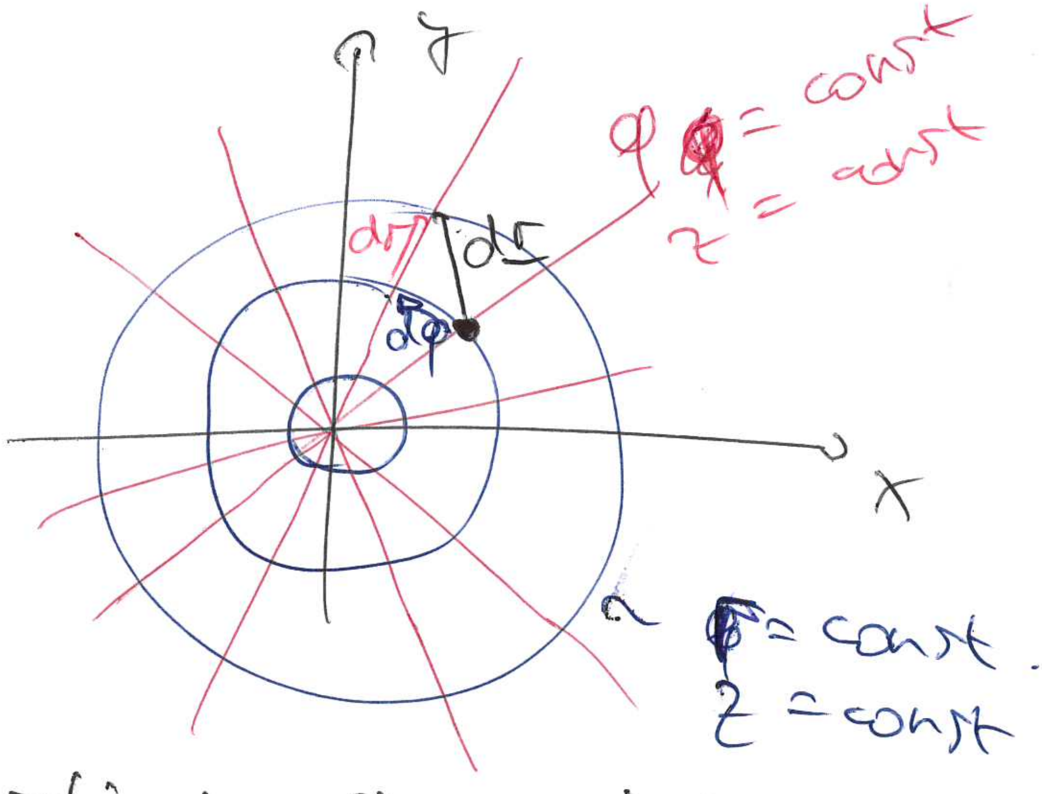
LECTURE 15

$$\underline{r} = \begin{pmatrix} x(u_1, u_2, u_3) \\ y(u_1, u_2, u_3) \\ z(u_1, u_2, u_3) \end{pmatrix}$$

<u>E.g.:</u> $u_1 = r$, $u_2 = \varphi$, $u_3 = z$

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$z = z$$



coordinate lines intersect at right angles

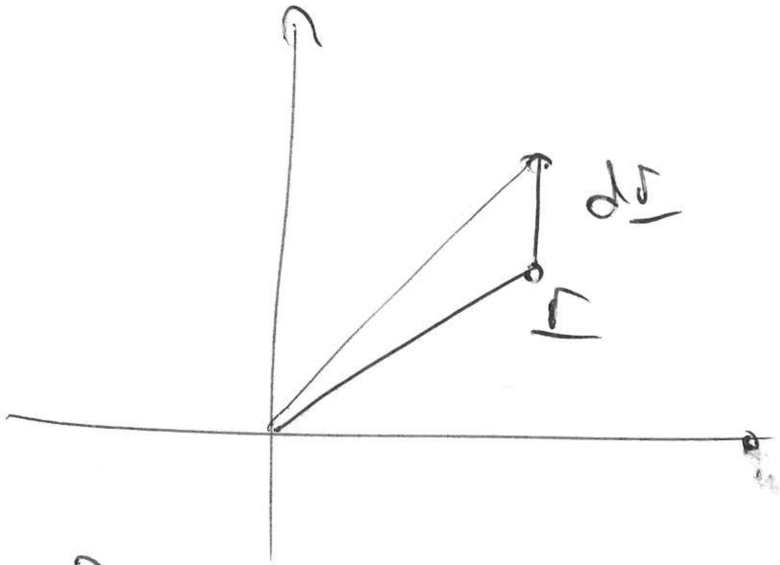
$\Longrightarrow$ orthogonal coord system.

As we change coordinates we move from

$\underline{r}(u_1, u_2, u_3)$ to

$\underline{r}(u_1 + du_1, u_2 + du_2, u_3 + du_3)$

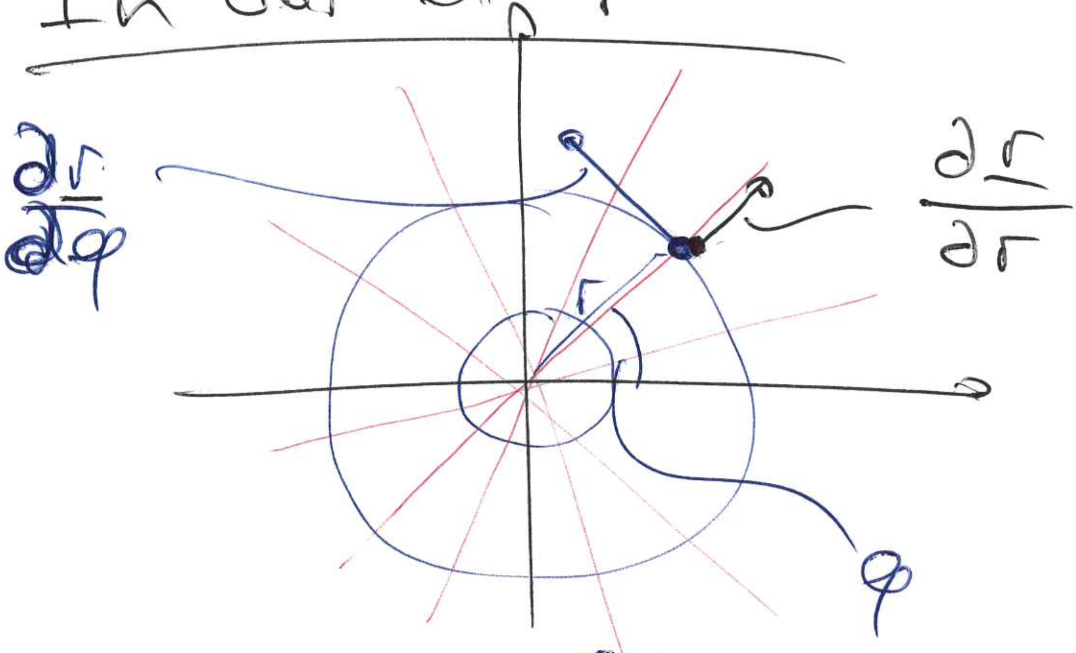
$= \underline{r}(u_1, u_2, u_3) + d\underline{r}$



$$d\underline{r} = \frac{\partial \underline{r}}{\partial u_1} du_1 + \frac{\partial \underline{r}}{\partial u_2} du_2 + \frac{\partial \underline{r}}{\partial u_3} du_3$$

Note $\dfrac{\partial \underline{r}}{\partial u_i}$ are tangent vectors to the coordinate lines

In our example



$$\underline{r}(r, \varphi, z) = \begin{pmatrix} r \cos\varphi \\ r \sin\varphi \\ z \end{pmatrix}$$

Definition:

$$h_1 = \left|\frac{\partial \underline{r}}{\partial u_1}\right| ; h_2 = \left|\frac{\partial \underline{r}}{\partial u_2}\right| ; h_3 = \left|\frac{\partial \underline{r}}{\partial u_3}\right|$$

are called the scale factors for u_1, u_2 & u_3 respectively.

Unit vectors in the coordinate directions can then be defined by

$$\underline{e}_1 = \frac{1}{h_1}\frac{\partial \underline{r}}{\partial u_1} ; \underline{e}_2 = \frac{1}{h_2}\frac{\partial \underline{r}}{\partial u_2} ; \underline{e}_3 = \frac{1}{h_3}\frac{\partial \underline{r}}{\partial u_3}$$

This also implies

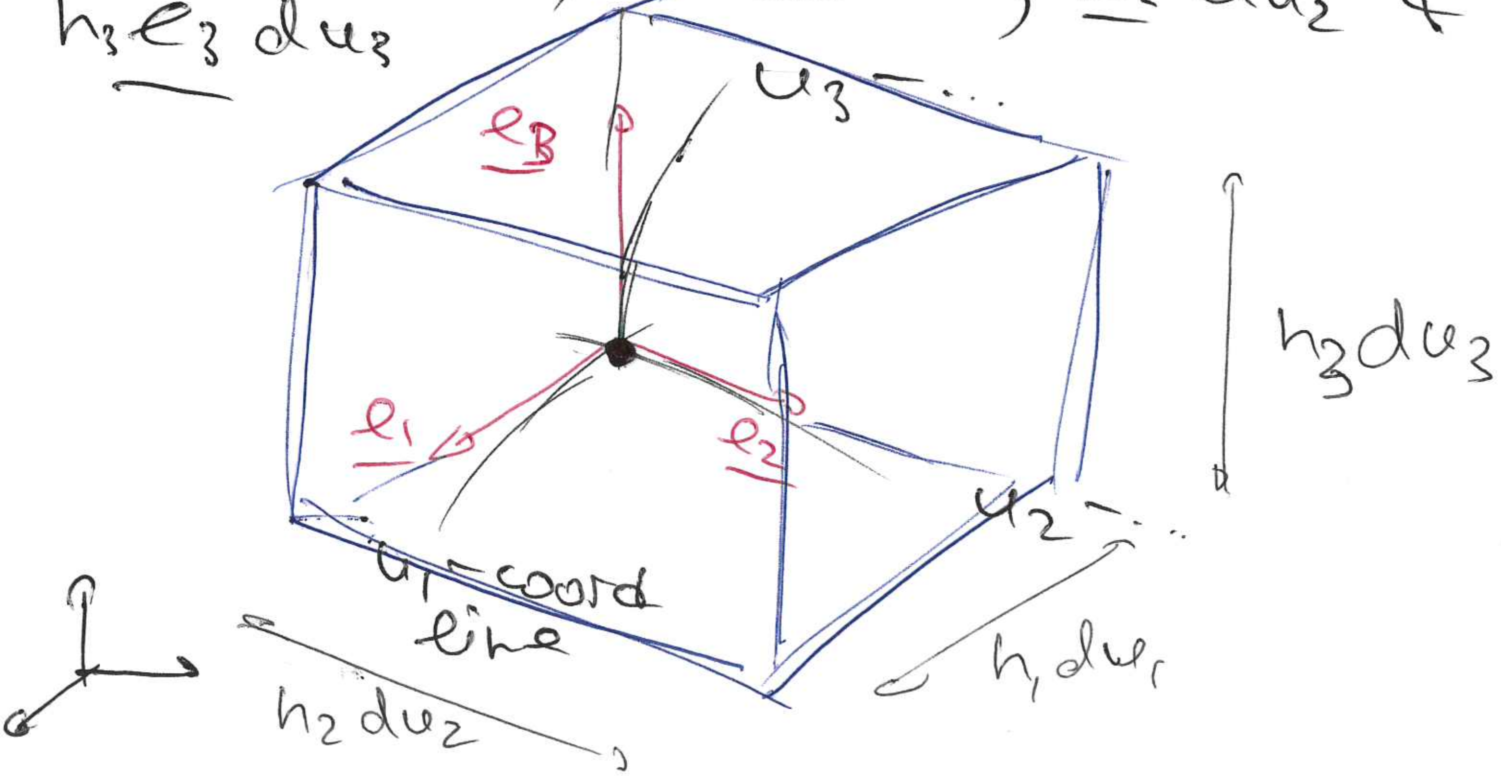
$$d\underline{r} = h_1 du_1 \underline{e_1} + h_2 du_2 \underline{e_2} + h_3 du_3 \underline{e_3}$$

For orthogonal coordinates the $\underline{e_i}$ vectors are orthonormal.

Arclength traversed as we increase (u_1, u_2, u_3) by (du_1, du_2, du_3) is

$$ds^2 = d\underline{r} \cdot d\underline{r} = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

Now consider the volume of the little parallelepiped spanned by $h_1 \underline{e_1} du_1$, $h_2 \underline{e_2} du_2$ & $h_3 \underline{e_3} du_3$



$$dV = h_1\, du_1 \cdot h_2\, du_2 \cdot h_3\, du_3$$

$$dV = \underbrace{(h_1 h_2 h_3)}_{\substack{\text{Jacobian of the} \\ \text{mapping between} \\ (x,y,z)\ \&\ (u_1,u_2,u_3)}} du_1\, du_2\, du_3$$

$$\left| \frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)} \right| = h_1 h_2 h_3$$

<u>Theorem:</u>

Relative to orthogonal curvilinear coordinates (u_1, u_2, u_3) grad, div & curl are given by

See pdf file (Lecture 17)

Procedure:

1. work out scale factors.
2. work out grad div & curl by using derivatives of ϕ or $\underline{F}$ w.r.t. u_1, u_2, u_3 & factors involving h_1, h_2, h_3.

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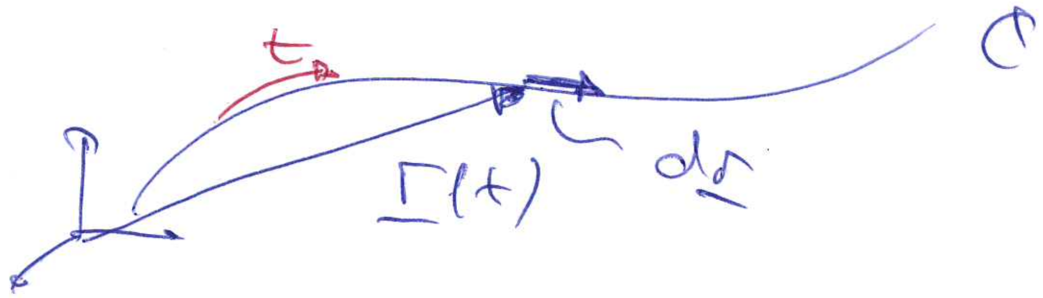
§4 Line, surface & vol. integrals

Tangential line integrals:

Given $\underline{F}$ a vector field we wish to evaluate

$$I = \int_C \underline{F} \cdot d\underline{r}$$



to evaluate this:

$$d\underline{r} = \frac{d\underline{r}}{dt}\,dt$$

$$I = \int \underline{F} \cdot \frac{d\underline{r}}{dt}\,dt$$

Example:

$$\underline{F} = \begin{pmatrix} x^2 y \\ x z \\ -2yz \end{pmatrix}$$

evaluate $\int_C \underline{F} \cdot d\underline{r}$

along path $\underline{r}(t) = \begin{pmatrix} 4t \\ 2t^2 \\ t^3 \end{pmatrix}$ where t varies between 0 & 1.

(path between $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$

so:

$$\frac{d\underline{r}}{dt} = \begin{pmatrix} 4 \\ 4t \\ 3t^2 \end{pmatrix} ; \underline{F} = \begin{pmatrix} 16t^2 \cdot 2t^2 \\ 4t \cdot t^3 \\ -2 \cdot 2t^2 t^3 \end{pmatrix}$$