LECTURE 14

Aside

Curl $\underline{u} = \nabla \times \underline{u}$

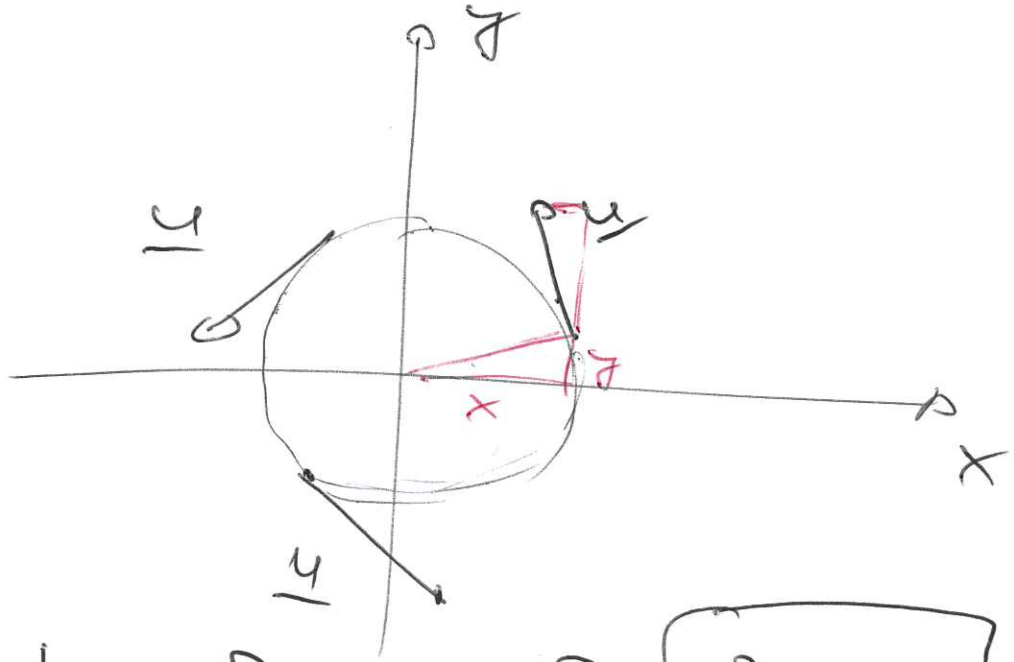
$\nabla \times \underline{u} = \underline{0} \Rightarrow$ irrotational

div $\underline{u} = \nabla \cdot \underline{u}$

$\nabla \cdot \underline{u} = 0 \Rightarrow$ solenoidal.

What is the relation between $\nabla \times \underline{u}$ & rotation?

Consider the velocity field due to rigid body rotation about z axis.



$$|\underline{u}| = \Omega r = \Omega \sqrt{x^2 + y^2}$$

angular veloc.

$$\Rightarrow \underline{u} = \Omega \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$ Direction ✓

Check magnitude

$$|\underline{u}| = \Omega \sqrt{x^2+y^2+0^2} = \Omega r$$ ✓

$$\text{curl } \underline{u} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \underbrace{\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}}_{\underline{u}}$$

$$\text{curl } \underline{u} = \begin{pmatrix} \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \\ \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \\ \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \end{pmatrix} = \Omega \begin{pmatrix} 0-0 \\ 0-0 \\ 1+1 \end{pmatrix}$$

$$\text{curl } \underline{u} = 2\Omega \, \underline{e_z}$$

So curl $\underline{u}$ represents the vectorial rotation.

Orthogonal curvilinear coordinates

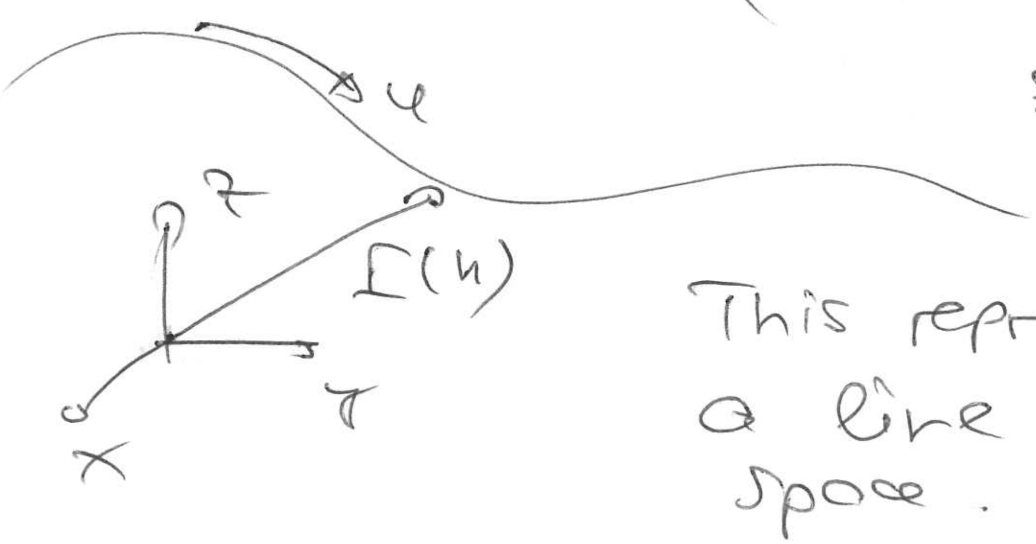
Earlier in course expressed $\nabla^2 \phi$ in different coordinate systems. Tedious!

We can ~~expr~~ obtain general expressions for grad, div, curl for general orthogonal curvilinear coords.

Reminder: Vector derivative

Vector $\underline{r}$ parametrised by scalar parameter u

$$\underline{r}(u) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix} = x(u)\,\underline{i} + y(u)\,\underline{j} + z(u)\,\underline{k}$$

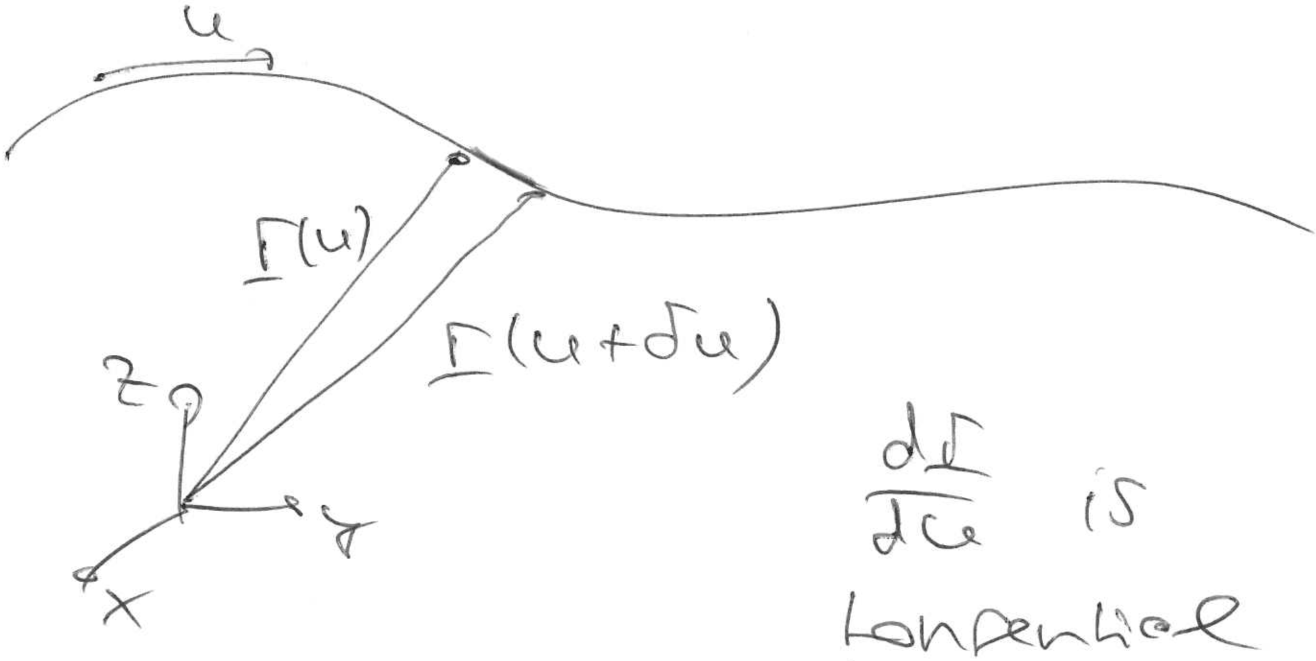


This represents a line in space.

Vector deriv:

$$\frac{d\underline{r}}{du} = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix}$$

$$\frac{d\underline{r}}{du} = \lim_{\delta u \to 0} \frac{\underline{r}(u+\delta u) - \underline{r}(u)}{\delta u}$$



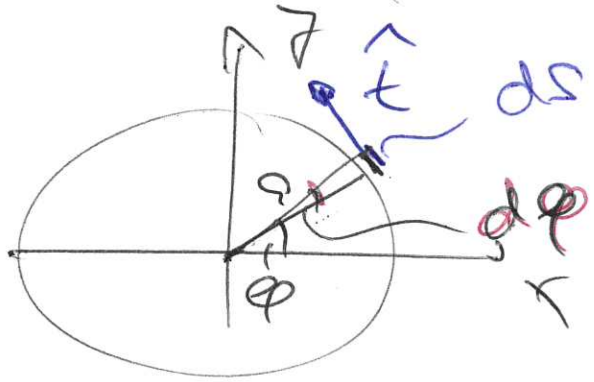
$\frac{d\underline{r}}{du}$ is tangential to the curve.

If $u = s =$ arclength then

$\frac{d\underline{r}}{ds} = \hat{\underline{t}}$ is a unit tangent to the curve.

Example: Circle radius a

$$\underline{r}(\varphi) = \begin{pmatrix} a\cos\varphi \\ a\sin\varphi \\ 0 \end{pmatrix}$$

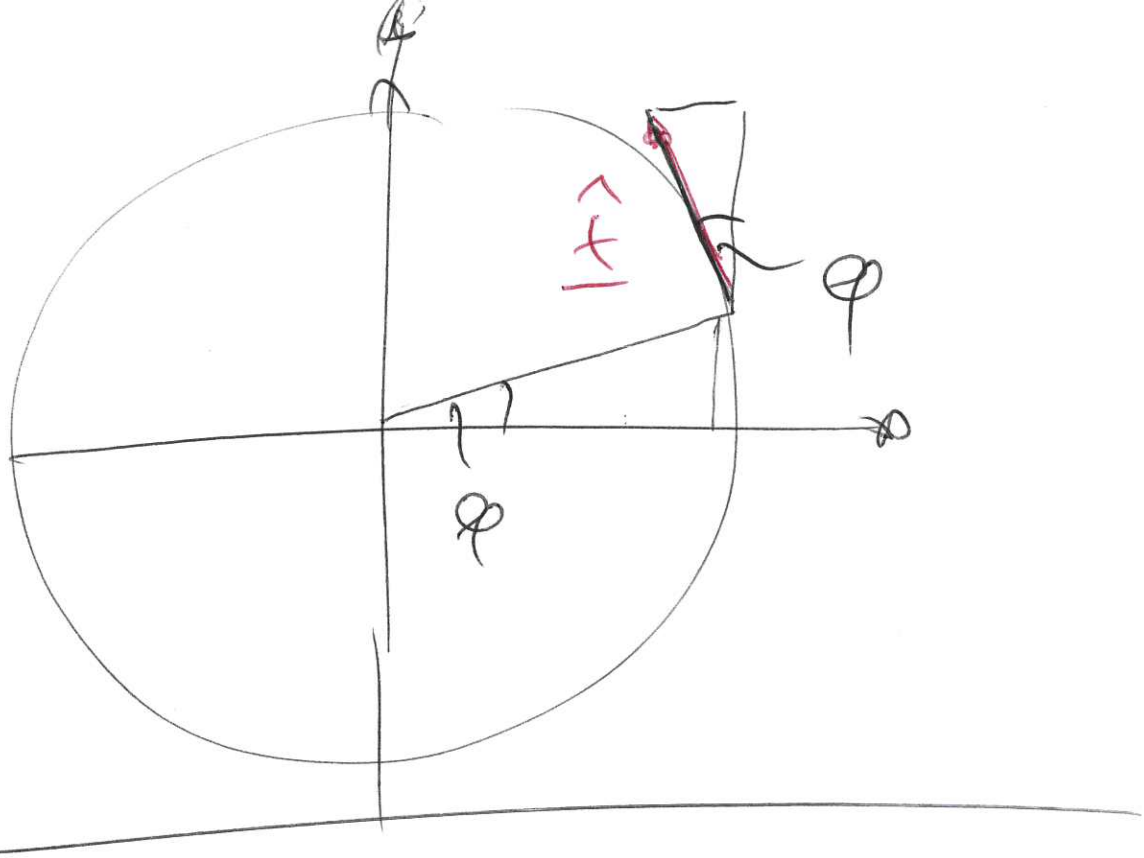


Can rewrite in terms of arclength: $ds = a\,d\varphi$

$$s = a\varphi \;;\; \varphi = \frac{s}{a}$$

$$\underline{r}(s) = \begin{pmatrix} a\cos\left(\frac{s}{a}\right) \\ a\sin\left(\frac{s}{a}\right) \\ 0 \end{pmatrix}$$

$$\frac{d\underline{r}}{ds} = \begin{pmatrix} -\sin\left(\frac{s}{a}\right) \\ \cos\left(\frac{s}{a}\right) \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$



Curvilinear coordinates are defined as a mapping between $(x, y, z) \Rightarrow (u_1, u_2, u_3)$

Assume that this mapping is one-to-one, continuous & has continuous first derivs.

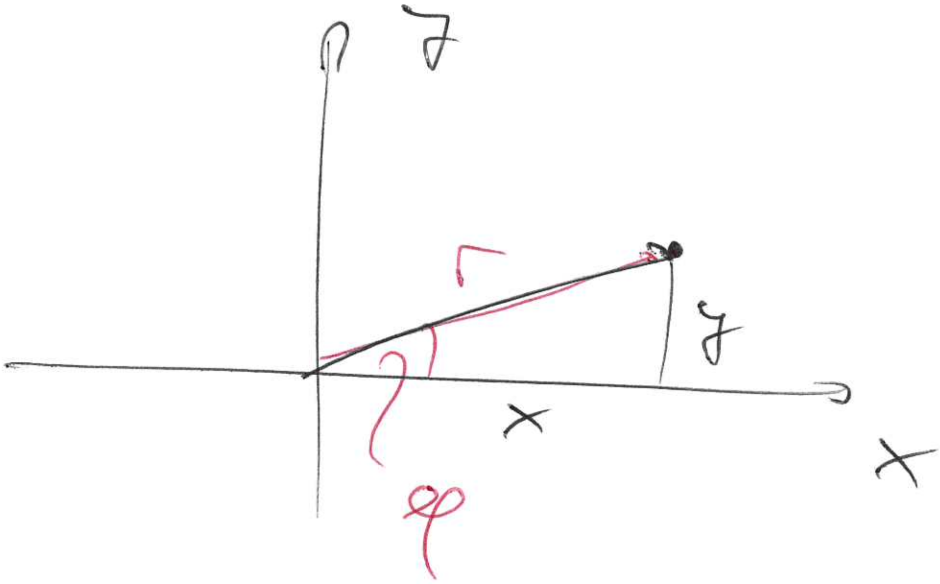
$x(u_1, u_2, u_3)$ $y(u_1, u_2, u_3)$ & $z(u_1, u_2, u_3)$

$\Rightarrow$ Any point P may be identified by exactly one triple of either (x, y, z) or (u_1, u_2, u_3)

Example: cylindrical polars:

$$u_1 = r \; ; \; u_2 = \varphi \; ; \; u_3 = z$$

$$x(u_1, u_2, u_3) = \cancel{x}(r, \varphi) = r\cos\varphi$$
$$y(u_1, u_2, u_3) = y(r, \varphi) = r\sin\varphi$$
$$z(u_1, u_2, u_3) = z \qquad = z$$

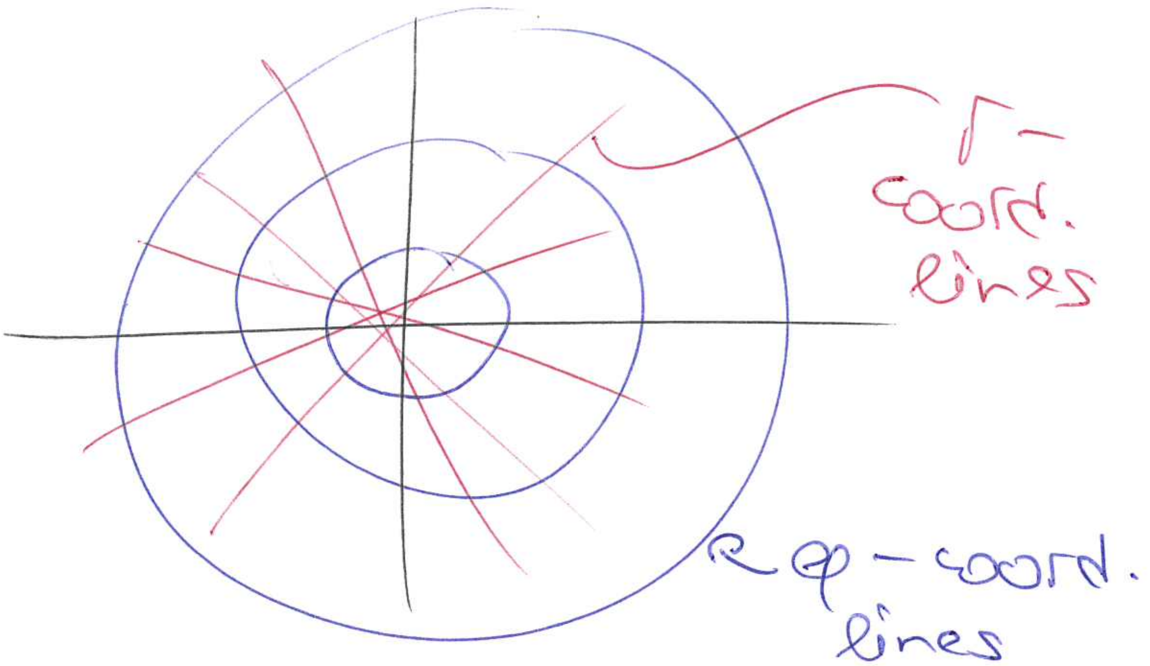


If u_1 is varied while keeping u_2 & u_3 constant we obtain a line in space: u_1 - coordinate curve.

In our example:

- u_1 (r) - coordinate lines are straight lines through origin.

• u_2 (φ) - coordinate lines are circles



Note: Coordinate lines intersect at a right angle.

This is what defines an orthogonal coord. system.