LECTURE 13

Proof:

$$\frac{d\phi}{ds} = \hat{\underline{d}} \cdot \nabla\phi = \underbrace{|\hat{\underline{d}}|}_{1} \, |\nabla\phi| \cos\theta$$

Choose $\hat{\underline{d}}$ to be tangential to the level surface S.

$\Rightarrow$ Along direction $\hat{\underline{d}}$ ϕ does not change (locally)

$$\Rightarrow \frac{d\phi}{ds} = 0 = \hat{\underline{d}} \cdot \nabla\phi$$

$\hat{\underline{d}}$ & $\nabla\phi$ are orthogonal

$\hookrightarrow$ tangential to level surface

$\Rightarrow$ normal to the level surface.

q.e.d.

Example: Sphere:

$$\phi(x,y,z) = x^2 + y^2 + z^2$$

(level surfaces of ϕ are spheres) $\phi(x,y,z) = c = R^2$

normal vector to these level surfaces:

$$\nabla\phi = 2x\,\underline{i} + 2y\,\underline{j} + 2z\,\underline{k}$$

Divergence

Given a vector field

$$\underline{F}(x,y,z) = F_x(x,y,z)\,\underline{i} + F_y(x,y,z)\,\underline{j} + F_z(x,y,z)\,\underline{k}$$

$$\operatorname{div}\underline{F} = \nabla\cdot\underline{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$$

$$\operatorname{div} \underline{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

is a scalar.

$\operatorname{div} \underline{F}$ is a linear operator because

$$\operatorname{div}(\alpha \underline{F} + \beta \underline{G}) =$$

$$\alpha \operatorname{div} \underline{F} + \beta \operatorname{div} \underline{G}$$

where α & β are constants.

Laplace operator

$$\operatorname{div}(\operatorname{grad} \phi) = \underbrace{\nabla \cdot \nabla}_{\nabla^2} \phi$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{pmatrix} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$\phi = x^2 y^2 z^2$

$$\nabla^2 \phi = \cancel{2 y^2 z^2 + 2 x^2 z^2 + 2 x^2 y^2}$$
$$= 2y^2 z^2 + 2x^2 z^2 + 2x^2 y^2$$

Curl

Given a vector field $\underline{F}(x, y, z)$

$$\text{curl } \underline{F} = \nabla \times \underline{F}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix}$$

curl $\underline{F}$ is a vector!

Def: If curl $\underline{F} = \underline{0}$

then $\underline{F}$ is called irrotational.

Second order operators and identities

Theorem:

$$\text{curl}(\text{grad}\,\phi)$$

$$= \underbrace{\nabla \times \nabla}\phi = 0.$$

If $\underline{F} = \nabla\phi$ then $\underline{F}$ is irrotational.

Proof: $f_x = \frac{\partial \phi}{\partial x}$ $\quad f_y = \frac{\partial \phi}{\partial y}$ $\quad f_z = \frac{\partial \phi}{\partial z}$

From above:

$$\nabla \times \underline{F} = \begin{pmatrix} \cancel{\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right)} \\ 0 \\ 0 \end{pmatrix}$$

Theorem:

$$\operatorname{div}(\operatorname{curl} \underline{F})$$

$$= \underbrace{\nabla \cdot \underbrace{(\nabla \times \underline{F})}_{\text{"orthogonal to } \nabla \text{"}}}_{0} = 0$$

Theorem:

$$\nabla \times (\nabla \times \underline{F}) = \operatorname{curl} \operatorname{curl} \underline{F} =$$

$$= \nabla (\nabla \cdot \underline{F}) - \nabla^2 \underline{F}$$

where

$$\nabla^2 \underline{F} = (\nabla^2 f_x)\,\underline{i} + (\nabla^2 f_y)\,\underline{j} + (\nabla^2 f_z)\,\underline{k}$$