LECTURE 12

Example: $\phi(x,y,z) = xy^2z^3$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}$$

$$= y^2z^3\underline{i} + 2xyz^3\underline{j} + 3xy^2z^2\underline{k}$$

∇ is a linear operator

$$\nabla(\alpha\phi(x,y,z) + \beta\psi(x,y,z)) =$$

$$\alpha\nabla\phi + \beta\nabla\psi$$

where α, β are const.

$$\nabla(\phi\psi) =$$

$$\frac{\partial}{\partial x}(\phi\psi)\underline{i} + \frac{\partial}{\partial y}(\phi\psi)\underline{j} + \frac{\partial}{\partial z}(\phi\psi)\underline{k}$$

$$= \left(\underline{\frac{\partial\phi}{\partial x}\psi} + \underline{\phi\frac{\partial\psi}{\partial x}}\right)\underline{i} + \left(\underline{\frac{\partial\phi}{\partial y}\psi} + \underline{\phi\frac{\partial\psi}{\partial y}}\right)\underline{j} +$$

$$\left(\underline{\frac{\partial\phi}{\partial z}\psi} + \underline{\phi\frac{\partial\psi}{\partial z}}\right)\underline{k}$$

$$= \phi \overbrace{\left(\frac{\partial \psi}{\partial x} \underline{i} + \frac{\partial \psi}{\partial y} \underline{j} + \frac{\partial \psi}{\partial z} \underline{k} \right)}^{\nabla \psi} +$$

$$+ \psi \underbrace{\left(\frac{\partial \phi}{\partial x} \underline{i} + \frac{\partial \phi}{\partial y} \underline{j} + \frac{\partial \phi}{\partial z} \underline{k} \right)}_{\nabla \phi}$$

$$\nabla(\phi\psi) = \phi \nabla \psi + \psi \nabla \phi$$

cf. product rule.

~~Exercise:~~ [illegible]

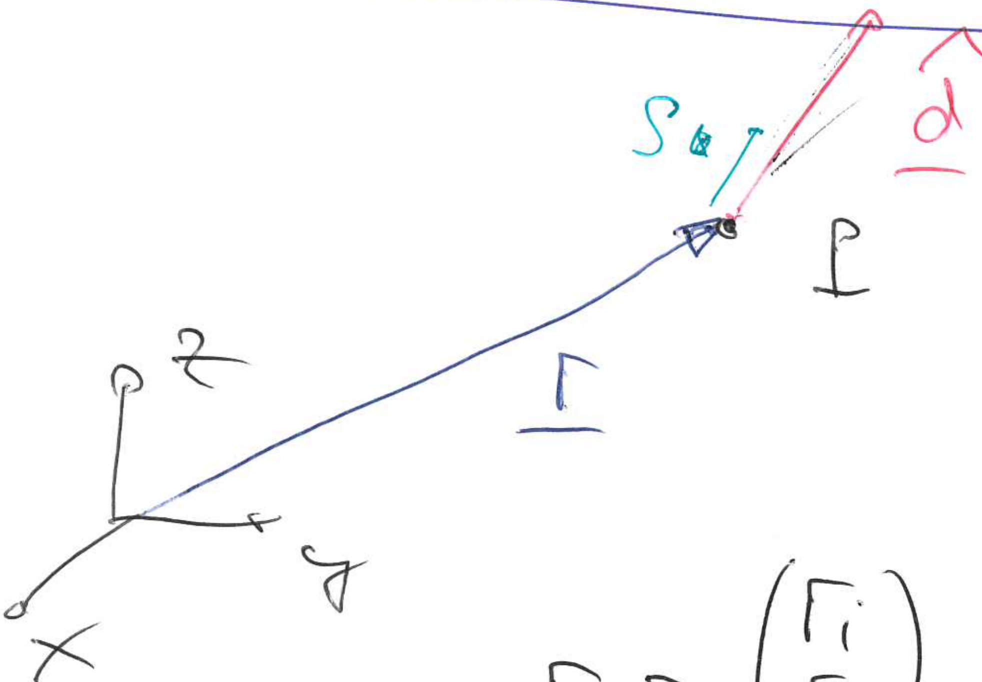
Directional derivative

$\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$ & $\frac{\partial \phi}{\partial z}$ represent the rate of change of ϕ in the direction of $\underline{i}$, $\underline{j}$ & $\underline{k}$

__Q__: What is the rate of change of ϕ at a point P in the direction of the unit vector $\underline{\hat{d}}$.

Def: Directional derivative:

$$\boxed{\left.\frac{d\phi}{ds}\right|_{\underline{r}} = \lim_{s \to 0} \frac{\phi(\underline{r} + s\hat{d}) - \phi(\underline{r})}{s}}$$



$$\underline{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \quad \underline{\hat{d}} = \begin{pmatrix} \hat{d}_1 \\ \hat{d}_2 \\ \hat{d}_3 \end{pmatrix}$$

$\phi(x, y, z)$

$$\left.\frac{d\phi}{ds}\right|_{\underline{r}} = \lim_{s \to 0} \frac{\phi(r_1 + s\hat{d}_1, r_2 + s\hat{d}_2, r_3 + s\hat{d}_3) - \phi(r_1, r_2, r_3)}{s}$$

$$\frac{d\phi}{ds} = \underline{\hat{d}} \cdot \nabla\phi$$

<u>Proof:</u>

$$\cancel{\frac{d\phi}{ds}}$$

$$\phi(x, y, z)$$

$$\phi(x(s), y(s), z(s))$$

$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds}$$

$$\begin{pmatrix} x(s) \\ y(s) \\ z(s) \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} + s \begin{pmatrix} \hat{d}_1 \\ \hat{d}_2 \\ \hat{d}_3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \\ \frac{dz}{ds} \end{pmatrix} = \begin{pmatrix} \hat{d}_1 \\ \hat{d}_2 \\ \hat{d}_3 \end{pmatrix}$$

$$\frac{d\phi}{ds} = \hat{d}_1 \frac{\partial\phi}{\partial x} + \hat{d}_2 \frac{\partial\phi}{\partial y} + \hat{d}_3 \frac{\partial\phi}{\partial z}$$

$$= \begin{pmatrix} \hat{d}_1 \\ \hat{d}_2 \\ \hat{d}_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{pmatrix}$$

$$\boxed{\frac{d\phi}{ds} = \hat{\underline{d}} \cdot \nabla \phi}$$

q.e.d.

Geometrical interpretation of $\nabla \phi$

If $\nabla \phi \neq \underline{0}$ then:

- $\nabla \phi$ points in the direction of the maximum increase in ϕ
- $|\nabla \phi|$ gives the max. rate of increase in that direction.

<u>Proof:</u> $\frac{d\phi}{ds} = \hat{d} \cdot \nabla \phi$

$$= \underbrace{|\hat{d}|}_{1} \cdot |\nabla \phi| \cos \Theta$$

where Θ is the angle between $\underline{\hat{d}}$ & $\nabla\phi$

Clearly $$\frac{d\phi}{ds} = |\nabla\phi| \cos\Theta$$

is max. if $\cos\Theta = 1$ or if $\Theta = 0$ so $\underline{\hat{d}}$ points in the direction of $\nabla\phi$.

q.e.d.

Normal to a surface

$$\phi(x,y,z) = C$$

C ← constant

defines a surface in 3D. (implicitly)

Example:

$$\phi(x,y,z) = x^2 + y^2 + z^2 = C$$

or $c = R^2$

$$x^2 + y^2 + z^2 = R^2$$

defines the surface of a sphere of radius $|R|$.

<u>Note:</u> could in principle solve $\phi(x,y,z) = c$ for $z = \phi^{-1}(x,y,c)$ but this is not always practical or helpful.

<u>Theorem:</u> $\nabla\phi$ is normal to the level surface of ϕ (where the level surface is formed by all the points satisfying $\phi(x,y,z) = c$)