$\boxed{\text{LECTURE 11}}$

Lagrange method

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$$

$$ds = \underbrace{\frac{dx}{a} = \frac{dy}{b}}_{y(x)\text{ for char}} = \frac{du}{c}$$

ONLY ALONG CHAR!

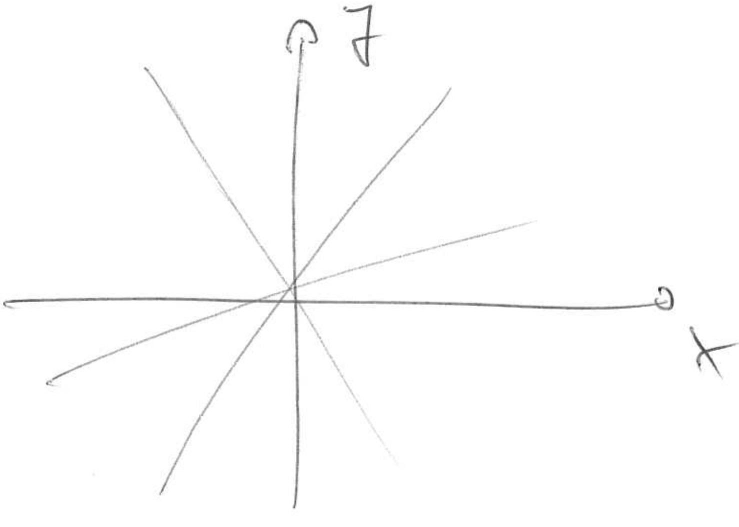
u

Example:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2xy$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{2xy}$$

$y(x) = Ax$ ← eqn for charact.



Note:

$A = \frac{y}{x}$

constant along each char.

Along char:

$$\frac{dx}{\cancel{x}} = \frac{du}{2\cancel{x}y}$$

$$\frac{du}{dx} = 2y$$ Only on char!

$y = Ax$

$$\frac{du}{dx} = 2Ax$$

$$u = Ax^2 + D$$

const along char.

some fct of x & y that is constant along the characteristic

$A = \frac{y}{x}$

$$u(x,y) = \frac{y}{\cancel{x}}x^2 + f\left(\frac{y}{x}\right)$$

$$u(x,y) = yx + f\left(\frac{y}{x}\right)$$

General soln.

Now apply IC:

If $y = 2x^2$: $u = 1$ (as before)

$$1 = u(x, 2x^2) = 2x^3 + f\left(\frac{2x^2}{x}\right)$$

$$\boxed{1 = 2x^3 + f(\underbrace{2x}_{\xi})}$$

$x = \frac{\xi}{2}$

$$1 = 2\left(\frac{\xi}{2}\right)^3 + f(\xi)$$

$$f(\xi) = 1 - \frac{1}{4}\xi^3$$

Specific soln:

$$u(x,y) = yx + 1 - \frac{1}{4}\left(\frac{y}{x}\right)^3$$

As before.

Example: Q4(i) Sheet 3

$$\boxed{\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \text{IC: } u(0,y) = \cos y}$$

$a = 1$

$b = -1$

$c = 0$

$$\underbrace{\frac{dx}{1} = -\frac{dy}{1}} = \underbrace{\frac{du}{0} = ds}$$

$$\frac{du}{ds} = 0$$

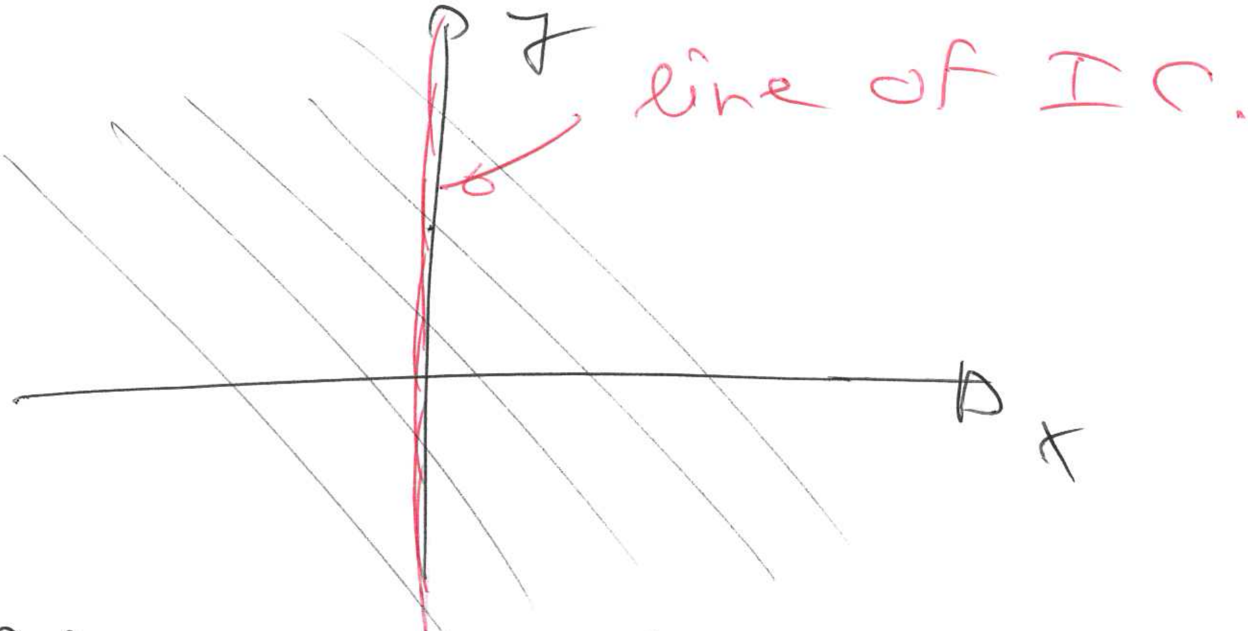
$$\Downarrow$$

$u = \text{const}$
along char.!

Char:

$$\frac{dy}{dx} = -1 \Rightarrow y = -x + A$$

A constant
along the char.



Along each ~~char~~ char.

$A = x + y = \text{const}.$

$\cancel{\frac{du}{d} = 0}$

$u = \text{const}$ along char.

$u = C$

& fct. of (x, y) provided ~~the~~ the fct stays constant along the charact.

$u = f(x + y)$

Gen. soln.

Apply IC: $u(x=0, y) = \cos y$

$f(y) = \cos y$

$f(\xi) = \cos(\xi)$

$\Rightarrow$ specific soln

$$u(x,y) = \cos(x+y)$$

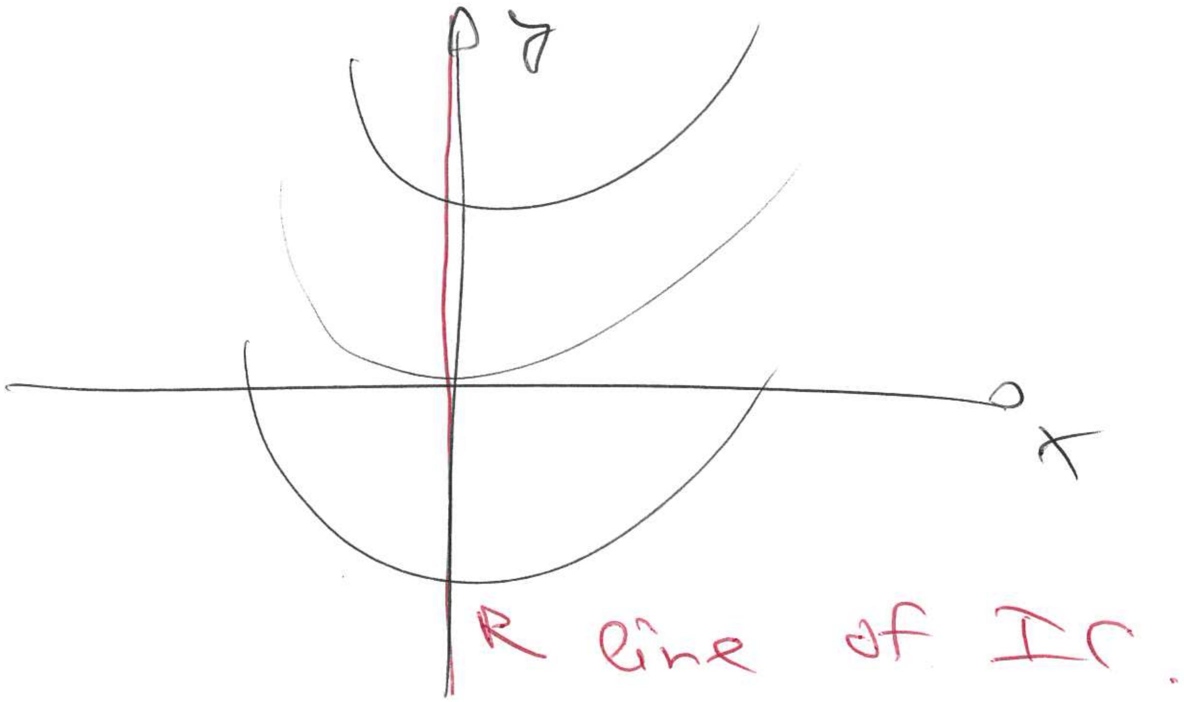
Example (Q4 (ii) sheet 3)

$$\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = u \qquad \text{IC: } u(0,y) = \sin y$$

$$\underbrace{\frac{dx}{1} = \frac{dy}{x}}_{} = \frac{du}{u}$$

$$\frac{dy}{dx} = x \Rightarrow y = \frac{1}{2}x^2 + A$$

$$A = y - \frac{1}{2}x^2 \quad \text{const along char.}$$



$$\frac{du}{u} = \frac{dx}{1}$$

$$\ln\left(\frac{u}{C'}\right) = x \qquad C' \text{ const along char.}$$

$$u = C' \exp(x)$$

$$u = f\left(y - \tfrac{1}{2}x^2\right) \exp(x)$$

Gen. soln.

IC: $u(x=0, y) = \sin y$

$$\sin(y) = f(y)$$

$$f(\xi) = \sin(\xi)$$

$$u(x,y) = \sin\left(y - \frac{1}{2}x^2\right)\exp(x)$$

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§3 Grad div & curl

Scalar fields:

$$\phi = \phi(x, y, z)$$ (e.g. temperature)

Vector fields

$$\underline{F} = \underline{F}(x, y, z)$$ (e.g. velocity)

Gradient of a scalar field

$$\text{grad}\,\phi = \nabla\phi =$$

$$\frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}$$

$$= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$ Nabla operator.