LECTURE 10

$$\boxed{\underbrace{x}_{a(x,y)}\frac{\partial u}{\partial x} + \underbrace{y}_{b(x,y)}\frac{\partial u}{\partial y} = \underbrace{2xy}_{c(x,y,u)}}$$

$$\left.\begin{aligned}\frac{dx}{ds} &= x \\ \frac{dy}{ds} &= y\end{aligned}\right\} \quad \begin{aligned}x(s) &= X e^{s} \\ y(s) &= Y e^{s}\end{aligned}$$

const. w.r.t s
constant for each characteristic

Along $(x(s), y(s))$:

$$\frac{du}{ds} = 2\,x(s)\,y(s) \Rightarrow u(s) = X Y e^{2s} + d$$

<u>IC</u>: $u = 1$ on $y = 2x^2$ / exclude $(x,y) = (0,0)$

$$\begin{pmatrix} X(r) \\ Y(r) \end{pmatrix} = \begin{pmatrix} r \\ 2r^2 \end{pmatrix}$$

$$\Rightarrow u(r,s) = 1 + 2r^3\left(e^{2s} - 1\right)$$

for
$$\boxed{\begin{aligned} x(r,s) &= r e^{s} \\ y(r,s) &= 2r^2 e^{s} \end{aligned}}$$

Parametric soln!

To turn this into $u(x,y)$ solve $x(r,s)$, $y(r,s)$ for $r(x,y)$ & $s(x,y)$

E.f: $\frac{y}{x} = 2r \Rightarrow r = \frac{y}{2x}$

Into $x(r,s)$:

$$x = re^s = \frac{y}{2x} e^s \; ; \; e^s = \frac{2x^2}{y}$$

$$s = \ln\left(\frac{2x^2}{y}\right)$$

into $u(r,s)$:

$$u = 1 + 2r^3\left(e^{2s} - 1\right)$$

$$u = 1 + 2\left(\frac{y}{2x}\right)^3\left(e^{2\ln\left(\frac{2x^2}{y}\right)} - 1\right)$$

$$u = 1 + 2\left(\frac{y}{2x}\right)^3\left(\left(\frac{2x^2}{y}\right)^2 - 1\right)$$

$$u(x,y) = 1 + xy - \frac{1}{4}\left(\frac{y}{x}\right)^3$$

Alternative:

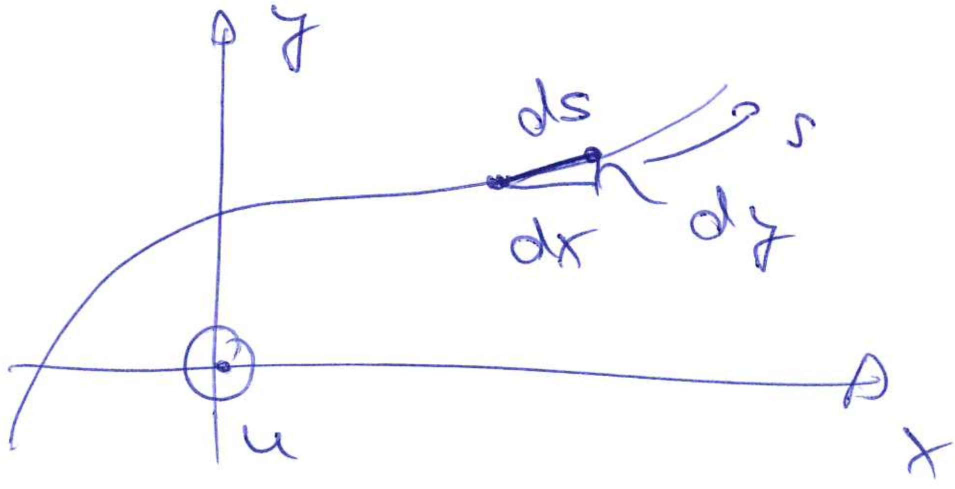
Lagrange's method

Recall: Along charact:

$$\frac{dx}{ds} = a(x,y)$$

$$\frac{dy}{ds} = b(x,y)$$

$$\frac{du}{ds} = c(x,y,u)$$



$$ds = \boxed{\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} = \frac{du}{c(x,y,u)}}$$

Combine 1st 2 eqns to

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$

First order ODE for $y(x)$ & i.c.

an explicit representation of the characteristic.

Solve

$y = G(x; C)$

↳ some (one!) constant of integration that identifies the charact. curve.

Trivial (?) observ: C is constant along that characteristic!

Now combine 1st & 3rd eqn:

$$\frac{dx}{a(x,y)} = \frac{du}{c(x,y,u)}$$

$$\frac{du}{dx} = \frac{c(x,y,u)}{a(x,y)}$$

along the charact.

⇓

i.e. for $y = G(x; C)$

$$\boxed{\frac{du}{dx} = \frac{c(x, y(x), u)}{a(x, y(x))}}$$

Another 1st order ODE for $u(x)$ (only on charact!)

Solve:

$$u = F(x; D)$$

↳ some (one!) other const. of integration. D is a number that is const. along that charact.

Done!?

Questions:
- where is y?
- D is constant w.r.t what?

Answer:
- soln only valid on charact. so $y = y(x)$

D is const. along the charact. but can vary with y.

$\Rightarrow$ D is a fct of (x, y) such that it is constant along the charact.

How do we find such a fct?

Recall $y = G(x; c)$

solve this for c

$c(x, y)$

$\Rightarrow$ $D = f(c(x, y))$

Example:

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2xy$$

revisited.

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{2xy}$$

First two eqns:

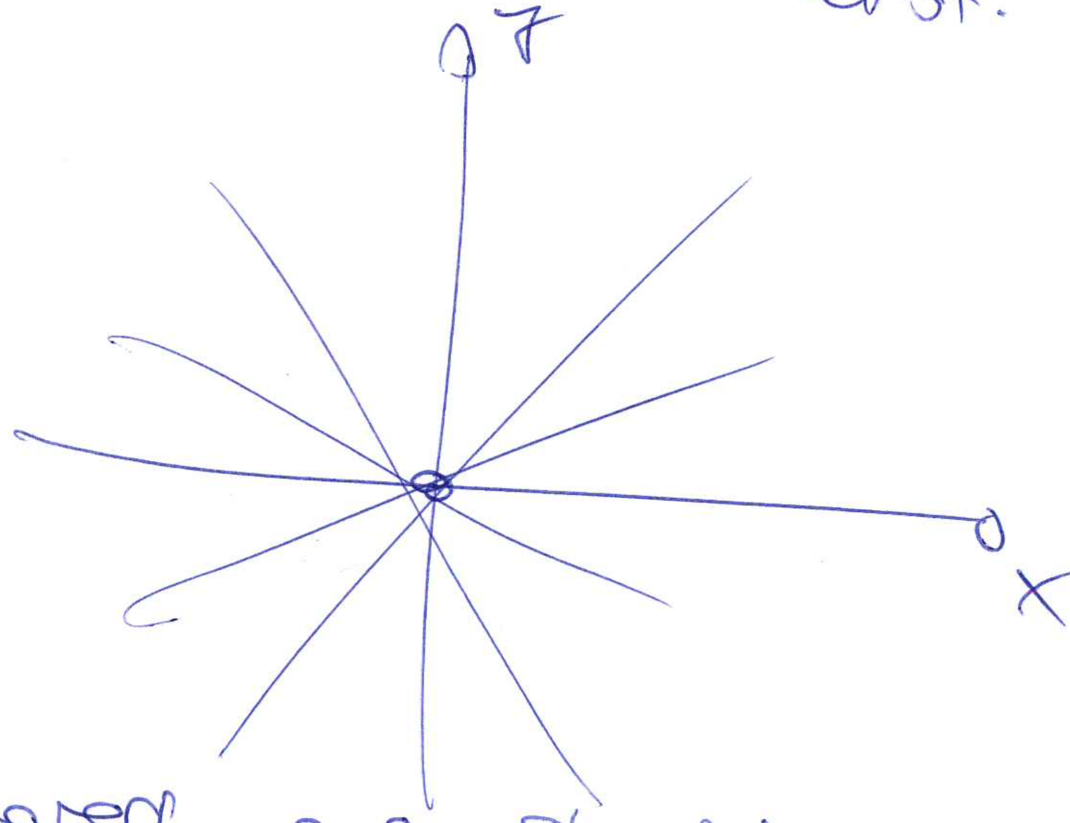
$$\frac{dy}{dx} = \frac{y}{x}$$ 1st order ODE for $y(x)$

Sep of vars:

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\ln \frac{y}{A} = \ln x$$

$$y(x) = Ax$$ for some const. A.



Charact. are straight lines through origin. (as before).