Lecture Notes for 2M1 – Q-Stream

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Course webpage:

http://www.maths.man.ac.uk/~mheil/Lectures/2M1

Note:

- This part of the course, dealing with functions of two variables and partial differential equations (PDEs), is taught from week 5.
- The course web page provides online access to the lecture notes, example sheets and other handouts and announcements.
- Most of the material will be taught in "chalk and talk" mode. If OHP transparencies are used, copies will be made available (after the lecture) on this page.
- Please consult the service course page

  http://www.maths.man.ac.uk/service

  for details on how to get hold of material for the other parts of the course.
- Please note that the lecture notes only summarise the main results and will generally be handed out *after* the material has been covered in the lecture. You are expected take notes during the classes.

1 Reminder: Functions of a single variable

A function $y = y(x)$ is a function of a single variable.

Examples

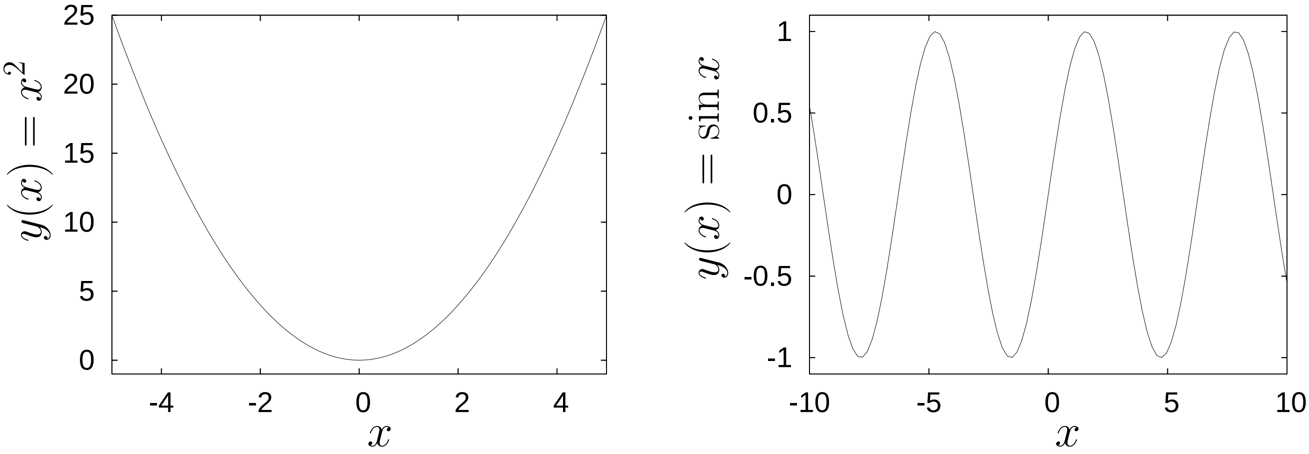


Figure 1: Functions of a single variable: $y(x) = x^2$ and $y(x) = \sin x$.

1.1 [Ordinary] derivatives

First derivative:

$$y' = \frac{dy}{dx}$$

- The first derivative represents the slope of the curve $y = y(x)$.
- In general, y' is a function of x too.

Second derivative:

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

- The second derivative is the derivative of the first derivative.
- The second derivative indicates the curvature of the curve $y = y(x)$.

Higher derivatives:

$$y^{(n)} = \frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\cdots\left(\frac{dy}{dx}\right)\cdots\right)\right)\right)$$

- Higher derivatives are defined recursively: The n-th derivative is the derivative of the $n-1$-th derivative.

1.2 Stationary points: Maxima and minima

Condition for a stationary point: .

- The function $y(x)$ has a "stationary point" at x_0 if

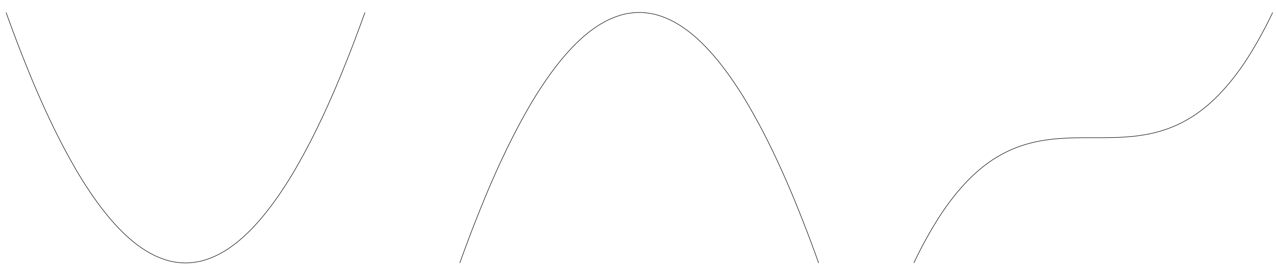
$$\left.\frac{dy}{dx}\right|_{x_0} = 0,$$

where the $(...)|_{x_0}$ notation indicates that the expression in the round brackets is to be evaluated at $x = x_0$.

Classification of stationary points: .

- The nature of a stationary point is determined by the function's second derivative:

$$\left.\frac{d^2y}{dx^2}\right|_{x_0} \begin{cases} > 0 & \Longrightarrow \quad \text{Local minimum} \\ < 0 & \Longrightarrow \quad \text{Local maximum} \\ = 0 & \Longrightarrow \quad \text{Test is not conclusive (curve too flat; e.g. at an inflection point.)} \end{cases}$$



(a) A local minimum (b) A local maximum (c) An inflection point

Figure 2: Generic stationary points for a function of one variable.

1.3 Taylor series:

- The Taylor series of a function $y(x)$ about a point $x = x_0$ provides an approximation of the function in the neighbourhood of x_0:

$$y(x_0+\epsilon) = y(x_0) + \left.\frac{dy}{dx}\right|_{x_0} \epsilon + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_0} \epsilon^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_0} \epsilon^3 + ...$$

  for "small" $|\epsilon|$.

  Here $n! = 1 \times 2 \times 3 \times .... \times (n-1) \times n$ is the factorial.

  The Taylor expansion may also be written as

$$y(x) = y(x_0) + \left.\frac{dy}{dx}\right|_{x_0} (x-x_0) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_0} (x-x_0)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_0} (x-x_0)^3 + ...$$

  for "small" values of $|x - x_0|$.

2 Functions of multiple [two] variables

In many applications in science and engineering, a function of interest depends on multiple variables. For instance, the ideal gas law $p = \rho RT$ states that the pressure p is a function of both its density, ρ and its temperature, T. (The gas constant R is a material property and not a variable).

We will now show how to extend the analysis of functions of a single variable to functions of multiple variables. We will restrict ourselves to the case of two variables, i.e. functions of the form

$$z = z(x, y),$$

the extension to larger numbers of variables being relatively straightforward, apart from the fact that functions of three and more variables are somewhat harder to visualise...

2.1 Examples

Here are some plots of functions of two variables:

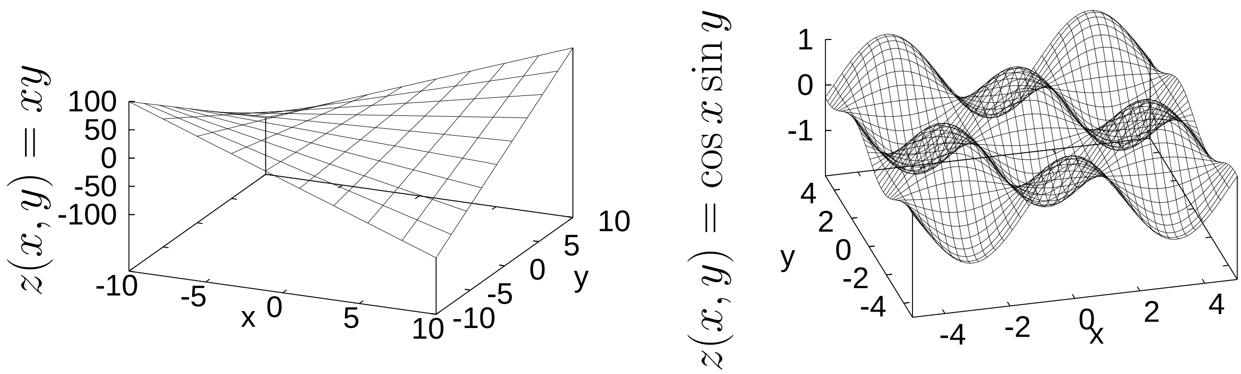


Figure 3: Functions of two variables: $z(x, y) = xy$ and $z(x) = \cos x \sin y$.

2.2 Partial derivatives

Functions of multiple variables can be differentiated with respect to either of their variables, the other variable being understood to be held constant during the differentiation. Such derivatives are known as *partial derivatives* and are distinguished from *ordinary derivatives* by using a ∂ instead of a d.

First derivatives: .

- For a function of two variables there are two partial derivatives
$$\frac{\partial z}{\partial x} = z_x$$
and
$$\frac{\partial z}{\partial y} = z_y.$$
- In general, the first derivatives are functions of x and y too.

Second derivatives: .

- For a function of two variables there are three second partial derivatives, defined as
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = z_{xx}$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = z_{yy}$$
and the mixed derivative
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = z_{xy}$$
where we usually[1] have
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$
i.e.
$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

Higher derivatives: .

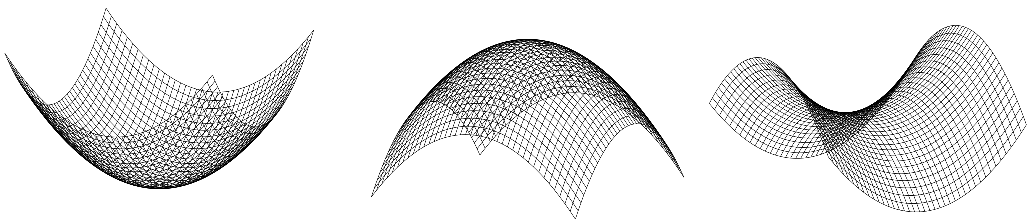
- Higher derivatives are again defined recursively, e.g.
$$\frac{\partial^5 z}{\partial x^3 \partial y^2} = \frac{\partial^3}{\partial x^3}\left(\frac{\partial^2 z}{\partial y^2}\right)$$
etc.

[1]Mathematicians obviously love to construct esoteric functions for which this property doesn't hold but it's rare to come across these in "real life". Strictly speaking, you can only exchange the order of the differentiation if the function $z(x, y)$ is sufficiently smooth. Most functions are.

2.3 Stationary points: Maxima and minima and saddles

Types of stationary points: .

- Functions of two variables can have stationary points of different types:



(a) A local minimum (b) A local maximum (c) A saddle point

Figure 4: Generic stationary points for a function of two variables.

Condition for a stationary point: .

- The function $z(x, y)$ has a "stationary point" at (x_0, y_0) if

$$\left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} = 0 \quad \underline{\textbf{and}} \quad \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)} = 0.$$

- This condition provides two equations for the two unknowns x_0 and y_0. These equations can have
  - No solution, in which case the function $z(x, y)$ has no stationary points.
  - A unique solution, in which case the function $z(x, y)$ has a single stationary point.
  - Multiple solutions, in which case the function $z(x, y)$ has multiple stationary points.

Classification of stationary points: .

- The nature of a stationary point is determined by the function's second derivatives. Here is a recipe for the classification of stationary points.

  For each stationary point (x_0, y_0):

  1. Determine the three second partial derivatives and evaluate them at the stationary point:
  $$A = \left.\frac{\partial^2 z}{\partial x^2}\right|_{(x_0,y_0)}, \quad B = \left.\frac{\partial^2 z}{\partial y^2}\right|_{(x_0,y_0)}, \quad \text{and} \quad C = \left.\frac{\partial^2 z}{\partial x \partial y}\right|_{(x_0,y_0)}.$$
  2. Compute the "discriminant"
  $$D = AB - C^2.$$
  3. Classify the stationary point according to the following cases:
  $$\begin{array}{lll} D < 0 & \Longrightarrow & \text{Saddle point} \\ D > 0 \text{ and } \partial^2 z/\partial x^2 > 0 \ \left[\text{or } \partial^2 z/\partial y^2 > 0\right] & \Longrightarrow & \text{Local minimum} \\ D > 0 \text{ and } \partial^2 z/\partial x^2 < 0 \ \left[\text{or } \partial^2 z/\partial y^2 < 0\right] & \Longrightarrow & \text{Local maximum} \\ D = 0 & \Longrightarrow & \text{Test is inconclusive} \end{array}$$

2.4 Taylor series:

2.4.1 The leading-order terms

- The Taylor series of a function $z(x, y)$ about a point (x_0, y_0) provides an approximation of the function in the neighbourhood of (x_0, y_0):

$$\begin{aligned} z(x_0+\epsilon, y_0+\delta) &= z(x_0, y_0) + \\ &+ \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} \epsilon + \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)} \delta + \\ &+ \frac{1}{2!}\left[\left.\frac{\partial^2 z}{\partial x^2}\right|_{(x_0,y_0)} \epsilon^2 + 2 \left.\frac{\partial^2 z}{\partial x \partial y}\right|_{(x_0,y_0)} \epsilon\, \delta + \left.\frac{\partial^2 z}{\partial y^2}\right|_{(x_0,y_0)} \delta^2 \right] + \cdots \end{aligned}$$

for "small" values of ϵ and δ.

As in the 1D case, this may also be written as

$$\begin{aligned} z(x, y) &= z(x_0, y_0) + \\ &+ \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} (x-x_0) + \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)} (y-y_0) + \\ &+ \frac{1}{2!}\left[\left.\frac{\partial^2 z}{\partial x^2}\right|_{(x_0,y_0)} (x-x_0)^2 + 2 \left.\frac{\partial^2 z}{\partial x \partial y}\right|_{(x_0,y_0)} (x-x_0)\,(y-y_0) + \left.\frac{\partial^2 z}{\partial y^2}\right|_{(x_0,y_0)} (y-y_0)^2 \right] + \\ &+ \cdots \end{aligned}$$

for "small" values of $|x - x_0|$ and $|y - y_0|$.

2.4.2 The general form of the 2D Taylor series

The general expression for the Taylor series in two variables may be written as

$$f(x-x_0, y-y_0) = \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \left.\frac{\partial^n f}{\partial x^{n-k} \partial y^k}\right|_{(x_0,y_0)} (x-x_0)^{n-k} (y-y_0)^k \right\},$$

where

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{(n-k)!\; k!}$$

are the binomial coefficients. Recall that the n binomial coefficients $\begin{pmatrix} n \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ 1 \end{pmatrix}, ..., \begin{pmatrix} n \\ n \end{pmatrix}$ may be obtained from the n-th row of Pascal's triangle:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

3 Example

We shall illustrate the various techniques by considering the function

$$z(x, y) = x^3 + 3y - y^3 - 3x.$$

Here is a sketch of the function:

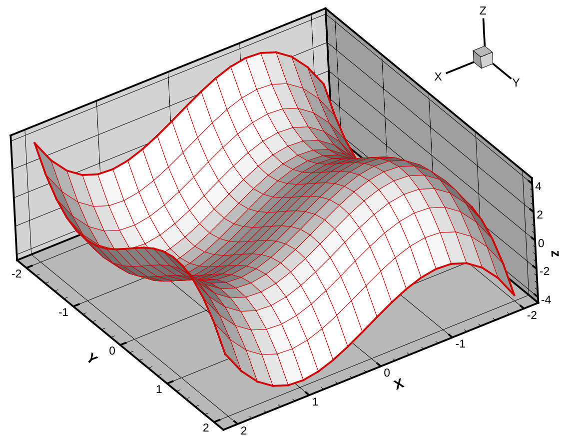


Figure 5: Sketch of the function $z(x, y) = x^3 + 3y - y^3 - 3x$.

Partial derivatives: .

- The partial derivatives are:

$$\frac{\partial z}{\partial x} = 3x^2 - 3.$$

$$\frac{\partial z}{\partial y} = 3 - 3y^2.$$

$$\frac{\partial^2 z}{\partial x^2} = 6x.$$

$$\frac{\partial^2 z}{\partial y^2} = -6y.$$

$$\frac{\partial^2 z}{\partial x \partial y} = 0.$$

Stationary points: .

- The coordinates (x_0, y_0) of any stationary points are given by the solution of the two equations:

$$\left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} = 3x_0^2 - 3 = 0$$

$$\left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)} = 3 - 3y_0^2 = 0.$$

[In the present example, these can be solved directly for $x_0 = \pm 1$ and $y_0 = \pm 1$, as the first equation only contains x_0 and the second one only y_0. In general, both equations will contain both unknowns and then have to be solved simultaneously.]

We therefore have four critical points:

$$\mathbf{P}_1 = (1,1) \quad \mathbf{P}_2 = (1,-1) \quad \mathbf{P}_3 = (-1,1) \quad \mathbf{P}_4 = (-1,-1).$$

- To classify the stationary points we evaluate the second derivatives in the following table:

Point	$A = 6x_0$	$B = -6y_0$	$C = 0$	$D = AB - C^2$	Classification
$\mathbf{P}_1 = (1,1)$	6	-6	0	-36	Saddle
$\mathbf{P}_2 = (1,-1)$	6	6	0	36	Local minimum
$\mathbf{P}_3 = (-1,1)$	-6	-6	0	36	Local maximum
$\mathbf{P}_4 = (-1,-1)$	-6	6	0	-36	Saddle

You should confirm these results by inspecting the plot shown at the beginning of the examples.

Taylor Expansion: .

- Finally, we determine the Taylor expansion of $z(x, y)$ about the point $(x, y) = (2, 1)$, where $z(2, 1) = 4$. We start by evaluating the derivatives:

$$\left.\frac{\partial z}{\partial x}\right|_{(2,1)} = 9$$

$$\left.\frac{\partial z}{\partial y}\right|_{(2,1)} = 0$$

$$\left.\frac{\partial^2 z}{\partial x^2}\right|_{(2,1)} = 12$$

$$\left.\frac{\partial^2 z}{\partial y^2}\right|_{(2,1)} = -6$$

$$\left.\frac{\partial^2 z}{\partial x \partial y}\right|_{(2,1)} = 0.$$

- In the vicinity of the point $(x, y) = (2, 1)$, i.e. for small values of ϵ and δ, the function $z(x, y)$ can therefore be approximated as

$$
\begin{aligned}
z(x = 2 + \epsilon, y = 1 + \delta) \;=\;& \underbrace{4}_{z(2,1)} + \underbrace{9}_{\frac{\partial z}{\partial x}\big|_{(2,1)}} \epsilon \; + \underbrace{0}_{\frac{\partial z}{\partial y}\big|_{(2,1)}} \delta \; + \\
&+ \; \frac{1}{2!}\left[\underbrace{12}_{\frac{\partial^2 z}{\partial x^2}\big|_{(2,1)}} \epsilon^2 \; + 2 \times \underbrace{0}_{\frac{\partial^2 z}{\partial x \partial y}\big|_{(2,1)}} \delta \, \epsilon \; + \underbrace{(-6)}_{\frac{\partial^2 z}{\partial y^2}\big|_{(2,1)}} \delta^2 \right] + ...
\end{aligned}
$$

So

$$y(x = 2 + \epsilon, y = 1 + \delta) = 4 + 9\epsilon + 6\epsilon^2 - 3\delta^2 + ...$$

or

$$y(x, y) = 4 + 9(x - 2) + 6(x - 2)^2 - 3(y - 1)^2 + ...$$

4 Reminder: Ordinary differential equations (ODEs)

- Ordinary differential equations (ODEs) are equations that relate the value of an unknown function of a single variable to its derivatives.

4.1 Examples

1. Equation of motion for a harmonic oscillator: The equation

$$m\frac{d^2x}{dt^2} + cx(t) = F(t)$$

is an ODE for the position $x(t)$ of a particle of mass m, mounted on a spring of stiffness c, when subjected to a time-dependent force $F(t)$. This is a second-order ODE because the highest derivative of the unknown function, $x(t)$, with respect to the independent variable, t, is of second order.

2. Transverse deflection of a string under axial tension: The equation

$$T\frac{d^2y}{dx^2} = p(x)$$

is an ODE that describes the transverse deflection $y(x)$ of a pre-stressed elastic string (under axial tension T), loaded transversely by a pressure $p(x)$. This is a second-order ODE because the highest derivative of the unknown function, $y(x)$, with respect to the independent variable, x, is of second order.

3. Radioactive decay: The equation

$$\frac{dm}{dt} = -\lambda m(t)$$

is an ODE that describes how the mass $m(t)$ of a radioactive material with decay rate λ decays. This is a first-order ODE because the highest derivative of the unknown function, $m(t)$, with respect to the independent variable, t, is of first first order.

4.2 Boundary and initial value problems

- ODEs must be augmented by additional constraints in the form of boundary or initial conditions. For second-order ODEs we can have either

  Boundary conditions: Boundary conditions specify the value of the unknown function at the "left" and "right" ends of the domain. The combination of an ODE and its boundary conditions is known as a boundary value problem. Boundary value problems typically arise in applications where the independent variable is a spatial coordinate, as in Problem 2 above. In this application it is "obvious" that the ODE (which describes the string's local equilibrium) must be augmented by the specification of the transverse deflection at the ends of the string – the string cannot just "float in space".

  or

  Initial conditions: Initial conditions specify they value of the unknown function and its first derivative at some "initial time". Initial value problems typically arise in applications where the independent variable is time, as in Problem 1 above. In this application it is "obvious" that the ODE (which describes the temporal evolution of the particle's position) must be augmented by the specification of its initial position, $x(t = 0)$, and its initial velocity, $dx/dt|_{t=0}$.

4.3 The solution of a boundary/initial value problem

- The solution to a boundary/initial value problem is *any* function that satisfies the ODE and the boundary/initial conditions.
- $\Longrightarrow$ It is easy to check if a function is a solution of a given boundary/initial value problem. However, it is not necessarily easy to find that solution from first principles.
- You have learned lots of techniques for the solution of the ODEs (separation of variables; integrating factor; ...) in your first year.

5 Partial differential equations (PDEs)

- Partial differential equations (PDEs) are functions that relate the value of an unknown function of multiple variables to its derivatives. In this course we will discuss four PDEs that arise in many science and engineering applications.
- For each PDE we will briefly discuss some of its physical background, the required boundary/initial conditions, and general properties of its solutions.
- Remember that, as in the case of ODEs, it is easy to check if a function is a solution of a given boundary/initial value problem. Simply check:

  1. Does the function satisfy the PDE?
  2. Does the function satisfy the boundary/initial conditions?

  If the answer to both tests is positive, the function is a solution.

- **Example:**

  Consider the boundary value problem comprising the PDE

  $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4$$

  in the unit disk $D = \Big\{(x, y) \;\Big|\; x^2 + y^2 \le 1\Big\}$, subject to the boundary condition

  $$u\Big|_{\partial D} = 1,$$

  where the domain boundary ∂D is given by $\partial D = \Big\{(x, y) \;\Big|\; x^2 + y^2 = 1\Big\}$.

  It is easy to verify that $u(x, y) = x^2 + y^2$ is a solution of the boundary value problem:

  1. Does the function satisfy the PDE?

     – Yes, because

     $$\frac{\partial^2 u}{\partial x^2} = 2 \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = 2,$$

     so

     $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4,$$

     as required.

  2. Does the function satisfy the boundary condition?

     – Yes, because

     $$u\Big|_{\partial D} = \left(x^2 + y^2\right)\Big|_{x^2+y^2=1} = 1,$$

     as required.

5.1 The 1D advection equation

5.1.1 The PDE and its applications

- The 1D advection equation

$$\frac{\partial u}{\partial t} + w(x,t)\,\frac{\partial u}{\partial x} = 0$$

  is a PDE for the unknown function $u(x,t)$. The equation arises in many transport processes where $u(x,t)$ represents, e.g. the concentration of a chemical that is advected by a one-dimensional flow field whose local velocity is given by the "wind" $w(x,t)$.

- The 1D advection equation requires an initial condition of the form

$$u(x, t=0) = u_0(x),$$

  where $u_0(x)$ is given.

- If the transport occurs in a finite domain, e.g. $x \in [X_L, X_R]$, and if $w(x,t) > 0$, a boundary condition of the form

$$u(x = X_L, t) = u_L(t),$$

  where $u_L(t)$ is given, must be specified. In the physical example referred to above this boundary condition specifies the concentration at the "inflow boundary".

5.1.2 Solution in an infinite domain for constant "wind"

- If the "wind" w is constant, the solution of the 1D advection equation has the form

$$u(x,t) = u_0(x - wt)$$

  where u_0 is the function that specifies the initial condition. This shows that the initial profile is simply "swept along" by the "wind" without changing its profile.

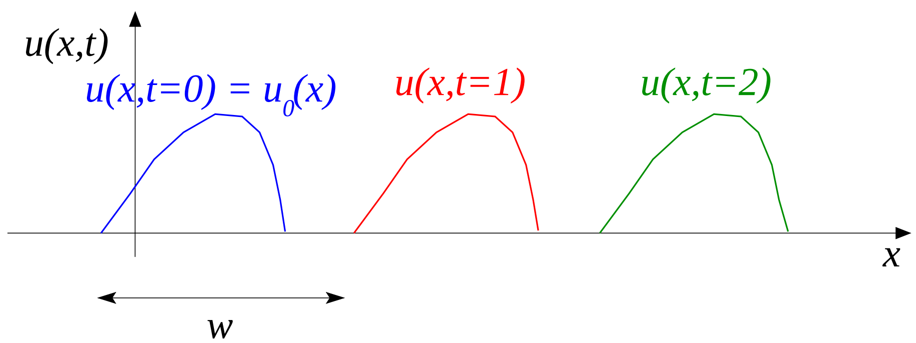


Figure 6: Solution of the 1D advection equation with constant wind. The initial profile $u(x, t = 0) = u_0(x)$ is "swept along" by the "wind" w.

5.2 The Laplace equation

- The Laplace equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
is a PDE for the unknown function $u(x, y)$, defined in a two-dimensional domain D.

- The PDE describes steady diffusion processes, and governs, for instance, the distribution of temperature in a block of material whose surface temperature is controlled.

- The 2D Laplace equation requires a boundary condition on all domain boundaries, i.e. the solution $u(x, y)$ must satisfy
$$u\big|_{\partial D} = u_0,$$
where the function u_0 is given. In the physical application referred to above, u_0 is the prescribed temperature distribution on the surface of the body.

5.3 The 1D unsteady heat equation

- The 1D unsteady heat equation
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$
is a PDE for the unknown function $u(x, t)$.

- The PDE describes unsteady diffusion processes, and governs, for instance, the spatial and temporal evolution of the temperature in a thin, well-insulated metal bar.

- The 1D unsteady heat equation requires an initial condition of the form
$$u(x, t = 0) = u_0(x)$$
where the function $u_0(x)$ is given.

- If solved in a finite domain, e.g. in the 1D domain $D = \big\{x \bigm| X_L \le x \le X_R\big\}$, we also require boundary conditions at both ends of the domain, i.e.
$$u(x = X_L, t) = u_L(t) \qquad \text{and} \qquad u(x = X_R, t) = u_R(t),$$
where the functions $u_L(t)$ and $u_R(t)$ are given.

- In the physical application referred to above, $u_0(x)$ describes the initial temperature distribution in the metal bar while $u_L(t)$ and $u_R(t)$ describe the prescribed temperature at its two ends.

5.4 The 1D linear wave equation

5.4.1 The PDE and its applications

- The 1D linear wave equation
$$\frac{\partial^2 u}{\partial t^2} = c^2 \, \frac{\partial^2 u}{\partial x^2}$$
is a PDE for the unknown function $u(x,t)$. The constant c is the "wave speed" whose role we shall discuss below.

- The PDE describes travelling-wave phenomena, and governs, for instance, the transverse displacements of an oscillating guitar string.

- The 1D linear wave equation requires two initial conditions, specifying the initial value and the initial time-derivative of the unknown function, respectively, i.e.
$$u(x, t=0) = u_0(x) \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = v_0(x)$$
where the functions $u_0(x)$ and $v_0(x)$ are given.

- If solved in a finite domain, e.g. in the 1D domain $D = \{x \mid X_L \le x \le X_R\}$, we also require boundary conditions at both ends of the domain, i.e.
$$u(x = X_L, t) = u_L(t) \quad \text{and} \quad u(x = X_R, t) = u_R(t),$$
where the functions $u_L(t)$ and $u_R(t)$ are given.

- In the physical application referred to above, $u_0(x)$ and $v_0(x)$ describe the initial position and the initial velocity of the guitar string, while the boundary conditions $u_L(t) = 0$ and $u_R(t) = 0$ indicate that the ends of the string are fixed to the rigid body of the guitar.

5.4.2 The general solution – travelling waves

- The general solution of the 1D linear wave equation has the form

$$u(x,t) = f(x - ct) + g(x + ct)$$

where f and g are arbitrary functions. The two functions represent two travelling waves, one moving to the right with speed c, the other one moving to the left with speed $-c$.

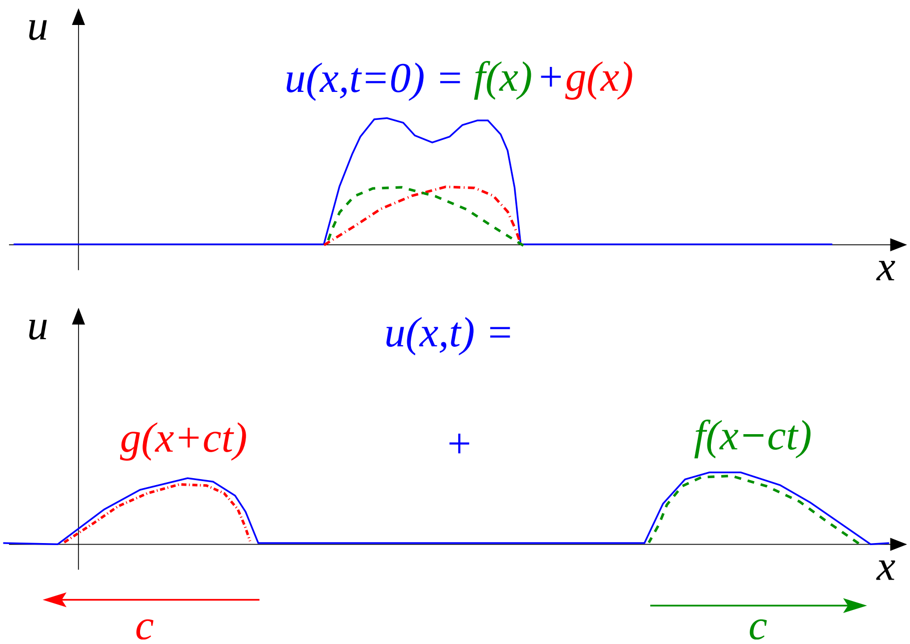


Figure 7: Solution of the 1D linear wave equation. The initial profile generates two travelling waves.

5.5 Solution of PDEs by separation of variables: Standing waves

One of the most powerful methods for the solution of PDEs is the method of the "separation of variables". Here is a step-by-step procedure, illustrated for the 1D linear wave equation for $u(x,t)$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

in the 1D domain $x \in [0,1]$, subject to the initial conditions

$$u(x, t=0) = \sin(\pi x),$$

and

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0,$$

and the boundary conditions

$$u(x=0,t) = 0 \qquad \text{and} \qquad u(x=1,t) = 0.$$

This corresponds to the case of oscillating string, initially deformed into a half-sine wave and released from rest at time $t = 0$. Note that, for simplicity, we only consider the case of unit wave-speed, $c = 1$.

5.5.1 A step-by-step guide to the method of separation of variables

Step 1: Write the unknown function of two variables as a product of two functions of a single variable:

$$u(x,t) = X(x)\ T(t).$$

This is an "ansatz" for the solution. Note that, in general, there is no a-priori guarantee that the solution can actually be written in this form but it's usually worth trying!

Step 2: Insert this "ansatz" into the PDE and differentiate.

$$X(x)\ \ddot{T}(t) = X''(x)\ T(t).$$

Note that

$$\ddot{T}(t) = \frac{d^2T}{dt^2}$$

and

$$T''(x) = \frac{d^2X}{dx^2}$$

are ordinary derivatives.

Step 3: Separate the variables, i.e. move all functions that only depend on t onto one side of the equation and all functions that depend only on x onto the other one:

$$\frac{\ddot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

Since the LHS now only depends on t and the RHS only on x, both must, in fact, be constant and we arbitrarily call the (as yet unknown) constant $-\omega^2$ to obtain

$$\frac{\ddot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = const. = -\omega^2$$

[See 5.5.3 for a more detailed discussion of this step.]

Step 4: Solve the "spatial" equation for $X(x)$

$$X''(x) + \omega^2 \; X(x) = 0 \quad \Longrightarrow \quad X(x) = \widehat{A}\sin(\omega x) + \widehat{B}\cos(\omega x)$$

for some constants $\widehat{A}$ and $\widehat{B}$. [Check your first-year lecture notes on techniques for solving constant-coefficient ODEs if this step is mysterious! In fact, you should know the solution of this ODE.]

Step 5: Apply the boundary conditions:

$$u(x=0,t) = X(0)\;T(t) = 0 \quad \Longrightarrow \quad X(0) = 0 \quad \Longrightarrow \quad \widehat{B} = 0.$$

$$u(x=1,t) = X(1)\;T(t) = 0 \quad \Longrightarrow \quad X(1) = 0 \quad \Longrightarrow \quad \widehat{A}\sin(\omega) = 0.$$

The latter equation can be satisfied either by setting $\widehat{A} = 0$ or $\omega = 0$ (in which case $u(x,t) \equiv 0$ which cannot satisfy the initial conditions) or by setting

$$\omega = \pi, \;\; 2\pi, \;\; 3\pi, ...$$

while leaving $\widehat{A}$ undetermined.

Step 6: Solve the "temporal equation" for $T(t)$:

$$\ddot{T}(t) + \omega^2 \; T(t) = 0 \quad \Longrightarrow \quad T(t) = \widehat{C}\sin(\omega t) + \widehat{D}\cos(\omega t)$$

for some constants $\widehat{C}$ and $\widehat{D}$.

Step 7: Combine the spatial and temporal factors and combine any superfluous undetermined constants:

$$u(x,t) = \widehat{A}\sin(\omega x)\big(\widehat{C}\sin(\omega t) + \widehat{D}\cos(\omega t)\big) = \sin(\omega x)\big(A\sin(\omega t) + B\cos(\omega t)\big)$$

where $A = \widehat{A}\widehat{C}$ and $B = \widehat{A}\widehat{D}$.

Step 8: Apply the initial conditions

$$u(x,t=0) = \sin(\pi x) = B\sin(\omega x) \quad \Longrightarrow \quad B = 1 \quad \text{and} \quad \omega = \pi.$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 = A\omega\sin(\omega x) \quad \Longrightarrow \quad A = 0.$$

Step 9: Done! The solution is

$$u(x,t) = \cos(\pi t)\sin(\pi x).$$

The oscillation of the string therefore represent a "standing wave" – the string oscillates between the two extrema $\pm\sin(\pi x)$ with a period of 2 time units, as shown in Fig. 8.

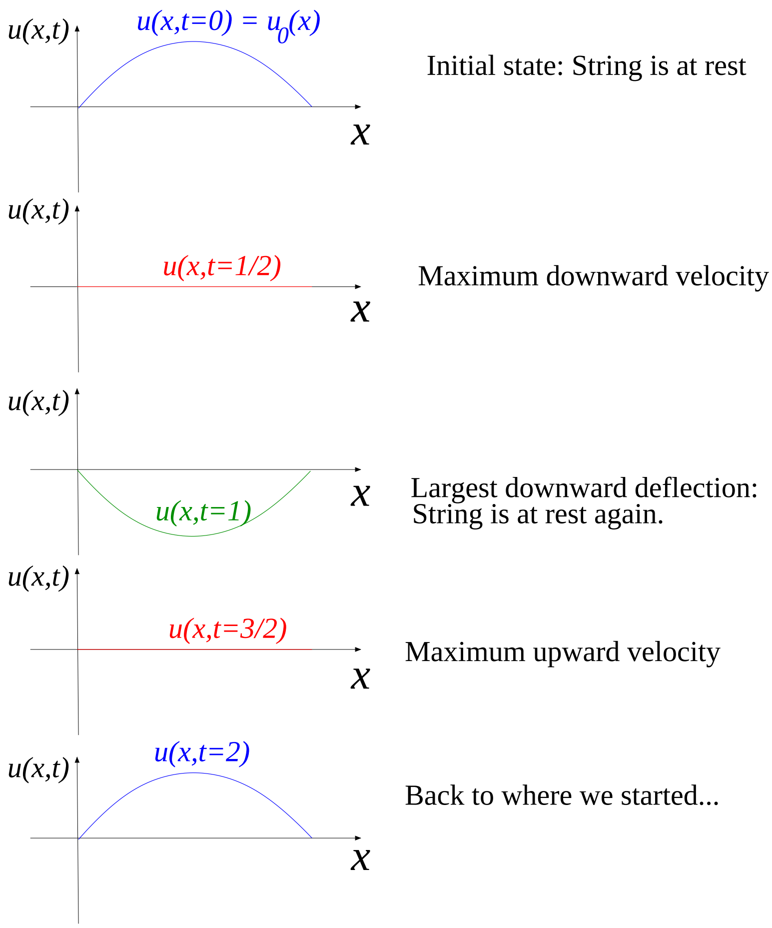


Figure 8: Solution of the 1D linear wave equation: A standing wave.

5.5.2 Comment 1: Relation to travelling waves

- The form of the solution obtained by the method of separation of variables may seem to contradict our claim regarding the form of the general solution made earlier. However, the two are equivalent: Using the trigonometric identity

$$2 \sin \alpha \cos \beta = \sin(\alpha - \beta) + \sin(\alpha + \beta)$$

with $\alpha = \pi x$ and $\beta = \pi t$ shows that

$$u(x,t) = \cos(\pi t) \sin(\pi x) = \frac{1}{2} \Big(\sin\big(\pi(x-t)\big) + \sin\big(\pi(x+t)\big) \Big),$$

consistent with our claim that the general solution has the form $u(x,t) = f(x-t) + g(x+t)$.

Standing waves can therefore be interpreted as the superposition of two travelling waves of identical shape, travelling in opposite directions.

5.5.3 Comment 2: The sign of the separation constant

- In **Step 3** of the separation of variables method, we had arbitrarily decided to give the (real) constant a negative value by writing it as $-\omega^2$. In the lecture we motivated this choice by our knowledge about the physics of the problem: We expect the string to oscillate periodically, so we wanted the ODE for $T(t)$ to have the form $\ddot{T} + \omega^2 T = 0$, rather than $\ddot{T} - \omega^2 T = 0$. What would have happened if had chosen the "wrong" sign?

If we had continued the analysis with the "wrong" sign we would soon have found that it is impossible to satisfy the boundary and initial conditions, forcing us to re-consider any ad-hoc choices made up to that point. Changing the sign of the separation constant is an easy way to "make the solution work". In general, a certain amount of trial and error may be required.

- However, it is important to remember that the method of separation of variables is not guaranteed to work for all problems!