

## 2M1 – Q-STREAM: SOLUTIONS <sup>1</sup> II

### 1. Solution of PDEs “by inspection”

- (a) To verify that the function  $u(x, y) = x - y$  is a solution of the PDE

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$$

we form the required partial derivatives

$$\frac{\partial u}{\partial x} = 1$$

and

$$\frac{\partial u}{\partial y} = -1,$$

showing that their sum is

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0,$$

as required.

- (b) To show that  $u(x, t) = \sin(x + t) + \cos(x - t)$  is a solution of the 1D linear wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

we form the required partial derivatives

$$\frac{\partial u}{\partial x} = \cos(x + t) - \sin(x - t)$$

(remember the chain rule!)

$$\frac{\partial^2 u}{\partial x^2} = -\sin(x + t) - \cos(x - t)$$

and

$$\frac{\partial u}{\partial t} = \cos(x + t) + \sin(x - t)$$

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<sup>1</sup>Any feedback to: *M.Heil@maths.man.ac.uk*

$$\frac{\partial^2 u}{\partial t^2} = -\sin(x+t) - \cos(x-t),$$

showing that the two second partial derivatives are identical, as required.

(c) We determine the required derivatives of  $u(x, t) = e^{at}(\sin x - bx^2)$ :

$$\frac{\partial u}{\partial t} = a e^{at}(\sin x - bx^2)$$

and

$$\frac{\partial^2 u}{\partial x^2} = e^{at}(-\sin x - 2b).$$

Inserting them into the 1D unsteady heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

yields

$$a e^{at}(\sin x - bx^2) = e^{at}(-\sin x - 2b).$$

This can be rewritten as

$$e^{at} [\sin x (a + 1) - bx^2 + 2b] = 0.$$

Since  $e^{at} \neq 0$ , the expression in the square brackets has to vanish for *all* values of the independent variable,  $x$ . This is only possible if the coefficients multiplying the various (linearly independent) functions vanish. This requires  $a = -1$  and  $b = 0$ .

## 2. Separation of variables for the 1D linear wave equation

We follow the procedure discussed in the lecture:

**Step 1:** Write the unknown function of two variables as a product of two functions of a single variable:

$$u(x, t) = X(x) T(t).$$

**Step 2:** Insert this “ansatz” into the PDE and differentiate.

$$X(x) \ddot{T}(t) = X''(x) T(t).$$

**Step 3:** Separate the variables, i.e. move all functions that only depend on  $t$  onto one side of the equation and all functions that depend only on  $x$  onto the other one:

$$\frac{\ddot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

Since the LHS now only depends on  $t$  and the RHS only on  $x$ , both must, in fact, be constant and we arbitrarily call the (as yet unknown) constant  $-\omega^2$  to obtain

$$\frac{\ddot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{const.} = -\omega^2$$

**Step 4:** Solve the “spatial” equation for  $X(x)$

$$X''(x) + \omega^2 X(x) = 0 \implies X(x) = \widehat{A} \sin(\omega x) + \widehat{B} \cos(\omega x)$$

for some constants  $\widehat{A}$  and  $\widehat{B}$ .

**Step 5:** Apply the boundary conditions:

$$u(x=0, t) = X(0) T(t) = 0 \implies X(0) = 0 \implies \widehat{B} = 0.$$

$$u(x=1, t) = X(1) T(t) = 0 \implies X(1) = 0 \implies \widehat{A} \sin(\omega) = 0.$$

The latter equation can be satisfied either by setting  $\widehat{A} = 0$  or  $\omega = 0$  (in which case  $u(x, t) \equiv 0$  which cannot satisfy the initial conditions) or by setting

$$\omega = \pi, 2\pi, 3\pi, \dots$$

while leaving  $\widehat{A}$  undetermined.

**Step 6:** Solve the “temporal equation” for  $T(t)$ :

$$\ddot{T}(t) + \omega^2 T(t) = 0 \implies T(t) = \widehat{C} \sin(\omega t) + \widehat{D} \cos(\omega t)$$

for some constants  $\widehat{C}$  and  $\widehat{D}$ .

**Step 7:** Combine the spatial and temporal factors and combine any superfluous undetermined constants:

$$u(x, t) = \widehat{A} \sin(\omega x) (\widehat{C} \sin(\omega t) + \widehat{D} \cos(\omega t)) = \sin(\omega x) (A \sin(\omega t) + B \cos(\omega t))$$

where  $A = \widehat{A}\widehat{C}$  and  $B = \widehat{A}\widehat{D}$ .

**Step 8:** Apply the initial conditions

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = \sin(3\pi x) = A\omega \sin(\omega x) \implies \omega = 3\pi \text{ and } A\omega = 1, \text{ i.e. } A = 1/(3\pi).$$

$$u(x, t = 0) = 0 = B \sin(\omega x) \implies B = 0.$$

**Step 9:** Done! The solution is

$$u(x, t) = \frac{1}{3\pi} \sin(3\pi t) \sin(3\pi x).$$

### 3. Separation of variables for the 1D unsteady heat equation

Recall that this problem may be interpreted as describing the spatial and temporal evolution of the temperature in a thin, well-insulated metal bar whose ends are held at zero temperature. Physically we expect the initial temperature distribution,  $u_0(x) = \sin(x)$ , to decay towards a state in which the temperature is zero everywhere.

The separation of variables method follows the same steps as in the linear wave example – the main difference being that we only have a first temporal derivative.

**Step 1:** Write the unknown function of two variables as a product of two functions of a single variable:

$$u(x, t) = X(x) T(t).$$

**Step 2:** Insert this “ansatz” into the PDE and differentiate.

$$X(x) \dot{T}(t) = X''(x) T(t).$$

**Step 3:** Separate the variables, i.e. move all functions that only depend on  $t$  onto one side of the equation and all functions that depend only on  $x$  onto the other one:

$$\frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

Since the LHS now only depends on  $t$  and the RHS only on  $x$ , both must, in fact, be constant and we arbitrarily call the (as yet unknown) constant  $-\omega^2$  to obtain

$$\frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{const.} = -\omega^2$$

**Step 4:** Solve the “spatial” equation for  $X(x)$

$$X''(x) + \omega^2 X(x) = 0 \implies X(x) = \widehat{A} \sin(\omega x) + \widehat{B} \cos(\omega x)$$

for some constants  $\widehat{A}$  and  $\widehat{B}$ .

**Step 5:** Apply the boundary conditions:

$$u(x = 0, t) = X(0) T(t) = 0 \implies X(0) = 0 \implies \widehat{B} = 0.$$

$$u(x = \pi, t) = X(\pi) T(t) = 0 \implies X(\pi) = 0 \implies \widehat{A} \sin(\omega \pi) = 0.$$

The latter equation can be satisfied either by setting  $\widehat{A} = 0$  or  $\omega = 0$  (in which case  $u(x, t) \equiv 0$  which cannot satisfy the initial conditions) or by setting

$$\omega = 1, 2, 3, \dots$$

while leaving  $\widehat{A}$  undetermined.

**Step 6:** Solve the “temporal equation” for  $T(t)$ :

$$\dot{T}(t) + \omega^2 T(t) = 0 \implies T(t) = \widehat{C} \exp(-\omega^2 t).$$

for some constant  $\widehat{C}$ . Note that if we had chosen another sign for the separation constant, the temperature in the metal bar would increase exponentially – not what we would expect!

**Step 7:** Combine the spatial and temporal factors and combine any superfluous undetermined constants:

$$u(x, t) = \widehat{A} \sin(\omega x) \widehat{C} \exp(-\omega^2 t) = A \sin(\omega x) \exp(-\omega^2 t)$$

where  $A = \widehat{A}\widehat{C}$ .

**Step 8:** Apply the initial conditions

$$u(x, t = 0) = \sin(x) = A \sin(\omega x) \implies A = 1 \quad \text{and} \quad \omega = 1$$

**Step 9:** Done! The solution is

$$u(x, t) = \exp(-t) \sin(x).$$

Compare to the result of question 1c.