5.5 Solution of PDEs by separation of variables: Standing waves

One of the most powerful methods for the solution of PDEs is the method of the "separation of variables". Here is a step-by-step procedure, illustrated for the 1D linear wave equation for $u(x,t)$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

in the 1D domain $x \in [0,1]$, subject to the initial conditions

$$u(x, t = 0) = \sin(\pi x),$$

and

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0,$$

and the boundary conditions

$$u(x = 0, t) = 0 \qquad \text{and} \qquad u(x = 1, t) = 0.$$

This corresponds to the case of oscillating string, initially deformed into a half-sine wave and released from rest at time $t = 0$. Note that, for simplicity, we only consider the case of unit wave-speed, $c = 1$.

5.5.1 A step-by-step guide to the method of separation of variables

Step 1: Write the unknown function of two variables as a product of two functions of a single variable:

$$u(x,t) = X(x)\ T(t).$$

This is an "ansatz" for the solution. Note that, in general, there is no a-priori guarantee that the solution can actually be written in this form but it's usually worth trying!

Step 2: Insert this "ansatz" into the PDE and differentiate.

$$X(x)\ \ddot{T}(t) = X''(x)\ T(t).$$

Note that

$$\ddot{T}(t) = \frac{d^2T}{dt^2}$$

and

$$T''(x) = \frac{d^2X}{dx^2}$$

are ordinary derivatives.

Step 3: Separate the variables, i.e. move all functions that only depend on t onto one side of the equation and all functions that depend only on x onto the other one:

$$\frac{\ddot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

Since the LHS now only depends on t and the RHS only on x, both must, in fact, be constant and we arbitrarily call the (as yet unknown) constant $-\omega^2$ to obtain

$$\frac{\ddot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = const. = -\omega^2$$

[See 5.5.3 for a more detailed discussion of this step.]

Step 4: Solve the "spatial" equation for $X(x)$

$$X''(x) + \omega^2 \; X(x) = 0 \quad \Longrightarrow \quad X(x) = \widehat{A}\sin(\omega x) + \widehat{B}\cos(\omega x)$$

for some constants $\widehat{A}$ and $\widehat{B}$. [Check your first-year lecture notes on techniques for solving constant-coefficient ODEs if this step is mysterious! In fact, you should know the solution of this ODE.]

Step 5: Apply the boundary conditions:

$$u(x=0,t) = X(0)\; T(t) = 0 \quad \Longrightarrow \quad X(0) = 0 \quad \Longrightarrow \quad \widehat{B} = 0.$$

$$u(x=1,t) = X(1)\; T(t) = 0 \quad \Longrightarrow \quad X(1) = 0 \quad \Longrightarrow \quad \widehat{A}\sin(\omega) = 0.$$

The latter equation can be satisfied either by setting $\widehat{A} = 0$ or $\omega = 0$ (in which case $u(x,t) \equiv 0$ which cannot satisfy the initial conditions) or by setting

$$\omega = \pi, \; 2\pi, \; 3\pi, ...$$

while leaving $\widehat{A}$ undetermined.

Step 6: Solve the "temporal equation" for $T(t)$:

$$\ddot{T}(t) + \omega^2 \; T(t) = 0 \quad \Longrightarrow \quad T(t) = \widehat{C}\sin(\omega t) + \widehat{D}\cos(\omega t)$$

for some constants $\widehat{C}$ and $\widehat{D}$.

Step 7: Combine the spatial and temporal factors and combine any superfluous undetermined constants:

$$u(x,t) = \widehat{A}\sin(\omega x)\big(\widehat{C}\sin(\omega t) + \widehat{D}\cos(\omega t)\big) = \sin(\omega x)\big(A\sin(\omega t) + B\cos(\omega t)\big)$$

where $A = \widehat{A}\widehat{C}$ and $B = \widehat{A}\widehat{D}$.

Step 8: Apply the initial conditions

$$u(x, t=0) = \sin(\pi x) = B\sin(\omega x) \quad \Longrightarrow \quad B = 1 \quad \text{and} \quad \omega = \pi.$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 = A\omega\sin(\omega x) \quad \Longrightarrow \quad A = 0.$$

Step 9: Done! The solution is

$$u(x,t) = \cos(\pi t)\sin(\pi x).$$

The oscillation of the string therefore represent a "standing wave" – the string oscillates between the two extrema $\pm\sin(\pi x)$ with a period of 2 time units, as shown in Fig. 8.

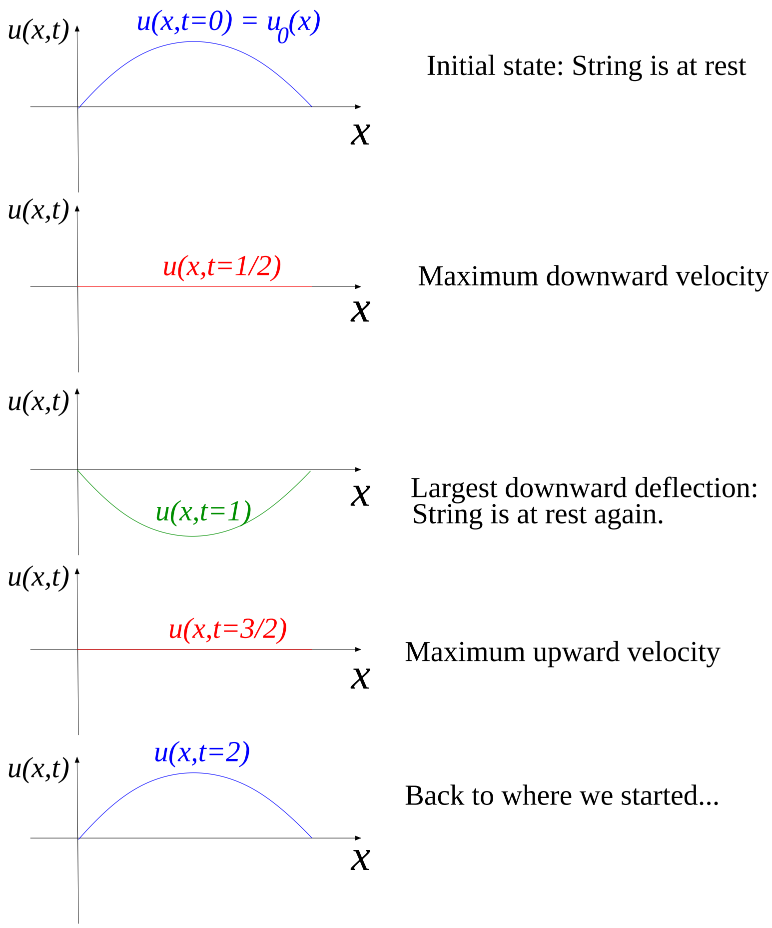

Figure 8: Solution of the 1D linear wave equation: A standing wave.

5.5.2 Comment 1: Relation to travelling waves

- The form of the solution obtained by the method of separation of variables may seem to contradict our claim regarding the form of the general solution made earlier. However, the two are equivalent: Using the trigonometric identity

$$2 \sin \alpha \cos \beta = \sin(\alpha - \beta) + \sin(\alpha + \beta)$$

with $\alpha = \pi x$ and $\beta = \pi t$ shows that

$$u(x,t) = \cos(\pi t) \sin(\pi x) = \frac{1}{2}\Big(\sin\big(\pi(x-t)\big) + \sin\big(\pi(x+t)\big)\Big),$$

consistent with our claim that the general solution has the form $u(x,t) = f(x-t) + g(x+t)$.

Standing waves can therefore be interpreted as the superposition of two travelling waves of identical shape, travelling in opposite directions.

5.5.3 Comment 2: The sign of the separation constant

- In **Step 3** of the separation of variables method, we had arbitrarily decided to give the (real) constant a negative value by writing it as $-\omega^2$. In the lecture we motivated this choice by our knowledge about the physics of the problem: We expect the string to oscillate periodically, so we wanted the ODE for $T(t)$ to have the form $\ddot{T} + \omega^2 T = 0$, rather than $\ddot{T} - \omega^2 T = 0$. What would have happened if had chosen the "wrong" sign?

If we had continued the analysis with the "wrong" sign we would soon have found that it is impossible to satisfy the boundary and initial conditions, forcing us to re-consider any ad-hoc choices made up to that point. Changing the sign of the separation constant is an easy way to "make the solution work". In general, a certain amount of trial and error may be required.

- However, it is important to remember that the method of separation of variables is not guaranteed to work for all problems!