5 Partial differential equations (PDEs)

- Partial differential equations (PDEs) are functions that relate the value of an unknown function of multiple variables to its derivatives. In this course we will discuss four PDEs that arise in many science and engineering applications.
- For each PDE we will briefly discuss some of its physical background, the required boundary/initial conditions, and general properties of its solutions.
- Remember that, as in the case of ODEs, it is easy to check if a function is a solution of a given boundary/initial value problem. Simply check:
  1. Does the function satisfy the PDE?
  2. Does the function satisfy the boundary/initial conditions?

  If the answer to both tests is positive, the function is a solution.
- **Example:**

  Consider the boundary value problem comprising the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4$$

  in the unit disk $D = \Big\{(x, y) \;\Big|\; x^2 + y^2 \le 1\Big\}$, subject to the boundary condition

$$u\Big|_{\partial D} = 1,$$

  where the domain boundary ∂D is given by $\partial D = \Big\{(x, y) \;\Big|\; x^2 + y^2 = 1\Big\}$.

  It is easy to verify that $u(x, y) = x^2 + y^2$ is a solution of the boundary value problem:

  1. Does the function satisfy the PDE?
     - Yes, because

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = 2,$$

       so

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4,$$

       as required.
  2. Does the function satisfy the boundary condition?
     - Yes, because

$$u\Big|_{\partial D} = \left(x^2 + y^2\right)\Big|_{x^2+y^2=1} = 1,$$

       as required.

5.1 The 1D advection equation

5.1.1 The PDE and its applications

- The 1D advection equation
$$\frac{\partial u}{\partial t} + w(x,t)\,\frac{\partial u}{\partial x} = 0$$
is a PDE for the unknown function $u(x,t)$. The equation arises in many transport processes where $u(x,t)$ represents, e.g. the concentration of a chemical that is advected by a one-dimensional flow field whose local velocity is given by the "wind" $w(x,t)$.

- The 1D advection equation requires an initial condition of the form
$$u(x, t = 0) = u_0(x),$$
where $u_0(x)$ is given.

- If the transport occurs in a finite domain, e.g. $x \in [X_L, X_R]$, and if $w(x,t) > 0$, a boundary condition of the form
$$u(x = X_L, t) = u_L(t),$$
where $u_L(t)$ is given, must be specified. In the physical example referred to above this boundary condition specifies the concentration at the "inflow boundary".

5.1.2 Solution in an infinite domain for constant "wind"

- If the "wind" w is constant, the solution of the 1D advection equation has the form
$$u(x,t) = u_0(x - wt)$$
where u_0 is the function that specifies the initial condition. This shows that the initial profile is simply "swept along" by the "wind" without changing its profile.

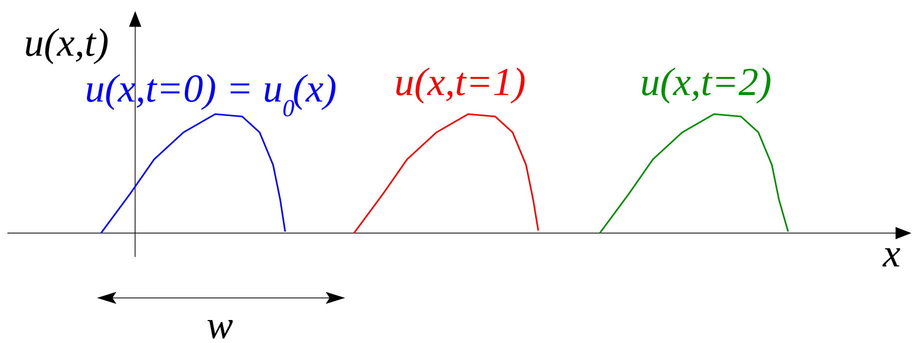

Figure 6: Solution of the 1D advection equation with constant wind. The initial profile $u(x, t = 0) = u_0(x)$ is "swept along" by the "wind" w.

5.2 The Laplace equation

- The Laplace equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
is a PDE for the unknown function $u(x, y)$, defined in a two-dimensional domain D.

- The PDE describes steady diffusion processes, and governs, for instance, the distribution of temperature in a block of material whose surface temperature is controlled.

- The 2D Laplace equation requires a boundary condition on all domain boundaries, i.e. the solution $u(x, y)$ must satisfy
$$u\big|_{\partial D} = u_0,$$
where the function u_0 is given. In the physical application referred to above, u_0 is the prescribed temperature distribution on the surface of the body.

5.3 The 1D unsteady heat equation

- The 1D unsteady heat equation
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$
is a PDE for the unknown function $u(x, t)$.

- The PDE describes unsteady diffusion processes, and governs, for instance, the spatial and temporal evolution of the temperature in a thin, well-insulated metal bar.

- The 1D unsteady heat equation requires an initial condition of the form
$$u(x, t = 0) = u_0(x)$$
where the function $u_0(x)$ is given.

- If solved in a finite domain, e.g. in the 1D domain $D = \{x \mid X_L \le x \le X_R\}$, we also require boundary conditions at both ends of the domain, i.e.
$$u(x = X_L, t) = u_L(t) \qquad \text{and} \qquad u(x = X_R, t) = u_R(t),$$
where the functions $u_L(t)$ and $u_R(t)$ are given.

- In the physical application referred to above, $u_0(x)$ describes the initial temperature distribution in the metal bar while $u_L(t)$ and $u_R(t)$ describe the prescribed temperature at its two ends.

5.4 The 1D linear wave equation

5.4.1 The PDE and its applications

- The 1D linear wave equation
$$\frac{\partial^2 u}{\partial t^2} = c^2 \, \frac{\partial^2 u}{\partial x^2}$$
is a PDE for the unknown function $u(x, t)$. The constant c is the "wave speed" whose role we shall discuss below.

- The PDE describes travelling-wave phenomena, and governs, for instance, the transverse displacements of an oscillating guitar string.

- The 1D linear wave equation requires two initial conditions, specifying the initial value and the initial time-derivative of the unknown function, respectively, i.e.
$$u(x, t = 0) = u_0(x) \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = v_0(x)$$
where the functions $u_0(x)$ and $v_0(x)$ are given.

- If solved in a finite domain, e.g. in the 1D domain $D = \{x \mid X_L \le x \le X_R\}$, we also require boundary conditions at both ends of the domain, i.e.
$$u(x = X_L, t) = u_L(t) \quad \text{and} \quad u(x = X_R, t) = u_R(t),$$
where the functions $u_L(t)$ and $u_R(t)$ are given.

- In the physical application referred to above, $u_0(x)$ and $v_0(x)$ describe the initial position and the initial velocity of the guitar string, while the boundary conditions $u_L(t) = 0$ and $u_R(t) = 0$ indicate that the ends of the string are fixed to the rigid body of the guitar.

5.4.2 The general solution – travelling waves

- The general solution of the 1D linear wave equation has the form

$$u(x,t) = f(x-ct) + g(x+ct)$$

where f and g are arbitrary functions. The two functions represent two travelling waves, one moving to the right with speed c, the other one moving to the left with speed $-c$.

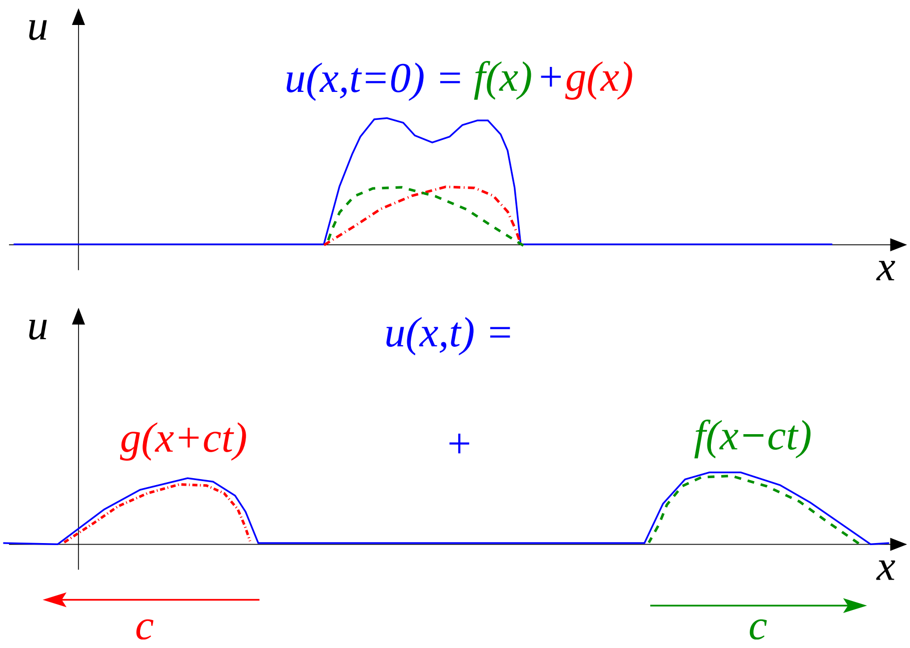

Figure 7: Solution of the 1D linear wave equation. The initial profile generates two travelling waves.