4 Reminder: Ordinary differential equations (ODEs)

- Ordinary differential equations (ODEs) are equations that relate the value of an unknown function of a single variable to its derivatives.

4.1 Examples

1. Equation of motion for a harmonic oscillator: The equation

$$m\frac{d^2x}{dt^2} + cx(t) = F(t)$$

is an ODE for the position $x(t)$ of a particle of mass m, mounted on a spring of stiffness c, when subjected to a time-dependent force $F(t)$. This is a second-order ODE because the highest derivative of the unknown function, $x(t)$, with respect to the independent variable, t, is of second order.

2. Transverse deflection of a string under axial tension: The equation

$$T\frac{d^2y}{dx^2} = p(x)$$

is an ODE that describes the transverse deflection $y(x)$ of a pre-stressed elastic string (under axial tension T), loaded transversely by a pressure $p(x)$. This is a second-order ODE because the highest derivative of the unknown function, $y(x)$, with respect to the independent variable, x, is of second order.

3. Radioactive decay: The equation

$$\frac{dm}{dt} = -\lambda m(t)$$

is an ODE that describes how the mass $m(t)$ of a radioactive material with decay rate λ decays. This is a first-order ODE because the highest derivative of the unknown function, $m(t)$, with respect to the independent variable, t, is of first first order.

4.2 Boundary and initial value problems

- ODEs must be augmented by additional constraints in the form of boundary or initial conditions. For second-order ODEs we can have either

  Boundary conditions: Boundary conditions specify the value of the unknown function at the "left" and "right" ends of the domain. The combination of an ODE and its boundary conditions is known as a boundary value problem. Boundary value problems typically arise in applications where the independent variable is a spatial coordinate, as in Problem 2 above. In this application it is "obvious" that the ODE (which describes the string's local equilibrium) must be augmented by the specification of the transverse deflection at the ends of the string – the string cannot just "float in space".

  or

  Initial conditions: Initial conditions specify they value of the unknown function and its first derivative at some "initial time". Initial value problems typically arise in applications where the independent variable is time, as in Problem 1 above. In this application it is "obvious" that the ODE (which describes the temporal evolution of the particle's position) must be augmented by the specification of its initial position, $x(t = 0)$, and its initial velocity, $dx/dt|_{t=0}$.

4.3 The solution of a boundary/initial value problem

- The solution to a boundary/initial value problem is *any* function that satisfies the ODE and the boundary/initial conditions.
- $\Longrightarrow$ It is easy to check if a function is a solution of a given boundary/initial value problem. However, it is not necessarily easy to find that solution from first principles.
- You have learned lots of techniques for the solution of the ODEs (separation of variables; integrating factor; ...) in your first year.