2M1 – Q-STREAM: SOLUTIONS [1] I

1. **Partial derivatives**

   (a) $f(x, y) = x^2 - 2xy + 6x - 2y + 1$:

$$\frac{\partial f}{\partial x} = 2x - 2y + 6$$

$$\frac{\partial f}{\partial y} = -2x - 2$$

$$\frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -2$$

$$\frac{\partial^2 f}{\partial y^2} = 0$$

   (b) $f(x, y) = \exp(xy)$:

$$\frac{\partial f}{\partial x} = y \ \exp(xy)$$

$$\frac{\partial f}{\partial y} = x \ \exp(xy)$$

$$\frac{\partial^2 f}{\partial x^2} = y^2 \exp(xy)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \exp(xy) + yx \ \exp(xy) = (1 + yx) \ \exp(xy)$$

$$\frac{\partial^2 f}{\partial y^2} = x^2 \exp(xy)$$

   (c) $f(x, y) = x^2 + y^2 + x^2 y + 4$:

$$\frac{\partial f}{\partial x} = 2x + 2xy$$

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$$\frac{\partial f}{\partial y} = 2y + x^2$$
$$\frac{\partial^2 f}{\partial x^2} = 2 + 2y$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2x$$
$$\frac{\partial^2 f}{\partial y^2} = 2$$

2. **Stationary points**

The position of stationary points is determined by the two conditions

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = 0 \quad \text{and} \quad \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = 0.$$

Their character is determined by the second derivatives; in particular the discriminant

$$D = AB - C^2,$$

where

$$A = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)}, \quad B = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)} \quad \text{and} \quad C = \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x_0,y_0)}.$$

(a) $f(x, y) = x^2 - 2xy + 6x - 2y + 1$:

Using the results from the previous question:

$$\left.\begin{array}{l} \partial f/\partial x\big|_{(x_0,y_0)} = 2x_0 - 2y_0 + 6 = 0 \\ \partial f/\partial y\big|_{(x_0,y_0)} = -2x_0 - 2 = 0 \end{array}\right\} \Longrightarrow (x_0, y_0) = (-1, 2).$$

$$A = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)} = 2$$
$$B = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)} = 0$$
$$C = \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x_0,y_0)} = -2$$

So

$$D = AB - C^2 = -4 < 0 \Longrightarrow \text{ Saddle point.}$$

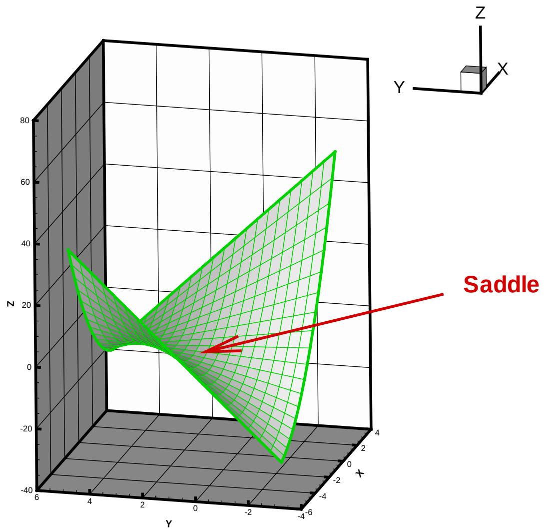

Figure 1: Plot of the function and its single stationary point.

(b) $f(x,y) = \exp(xy)$:

Using the results from the previous question:

$$\left.\begin{array}{l} \partial f/\partial x\big|_{(x_0,y_0)} = y_0 \ \exp(x_0 y_0) \\ \partial f/\partial y\big|_{(x_0,y_0)} = x_0 \ \exp(x_0 y_0) \end{array}\right\} \Longrightarrow (x_0, y_0) = (0,0).$$

$$A = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)} = 0$$

$$B = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)} = 0$$

$$C = \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x_0,y_0)} = 1$$

So

$$D = AB - C^2 = -1 < 0 \Longrightarrow \text{ Saddle point.}$$

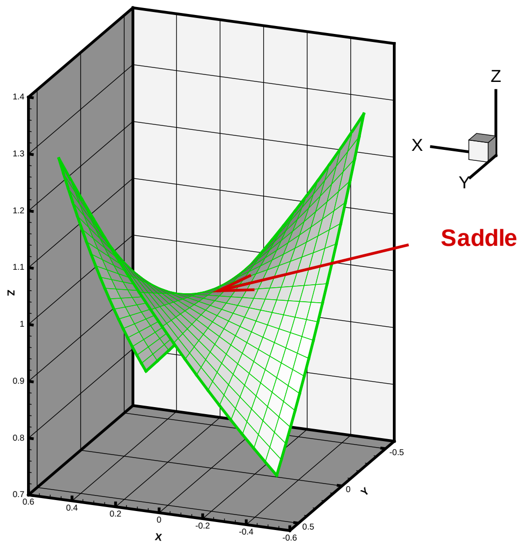

Figure 2: Plot of the function and its single stationary point.

(c) $f(x, y) = x^2 + y^2 + x^2 y + 4$:

Using the results from the previous question:

$$\partial f/\partial x\big|_{(x_0,y_0)} = 2x_0 + 2x_0 y_0 = 0$$
$$\partial f/\partial y\big|_{(x_0,y_0)} = 2y_0 + x_0^2 = 0$$

The second of these equations can be solved for $y_0 = -\frac{1}{2}x_0^2$. Inserting this into the first equation yields

$$2x_0 - x_0^3 = x_0(2 - x_0^2) = 0.$$

This equation has three solutions, corresponding to the three stationary points:

$$\mathbf{P}_1 : \quad x_0 = 0, \ y_0 = 0,$$
$$\mathbf{P}_2 : \quad x_0 = \sqrt{2}, \ y_0 = -1,$$

and

$$\mathbf{P}_3: \quad x_0 = -\sqrt{2},\ y_0 = -1.$$

As in the lecture, we analyse the character of the three critical point in a table:

Point	$A = 2 + 2y_0$	$B = 2$	$C = 2x_0$	$D = AB - C^2$	Classification
$\mathbf{P}_1 = (0,0)$	2	2	0	4	Local minimum
$\mathbf{P}_2 = (\sqrt{2}, -1)$	0	2	$2\sqrt{2}$	-8	Saddle
$\mathbf{P}_3 = (-\sqrt{2}, -1)$	0	2	$-2\sqrt{2}$	-8	Saddle

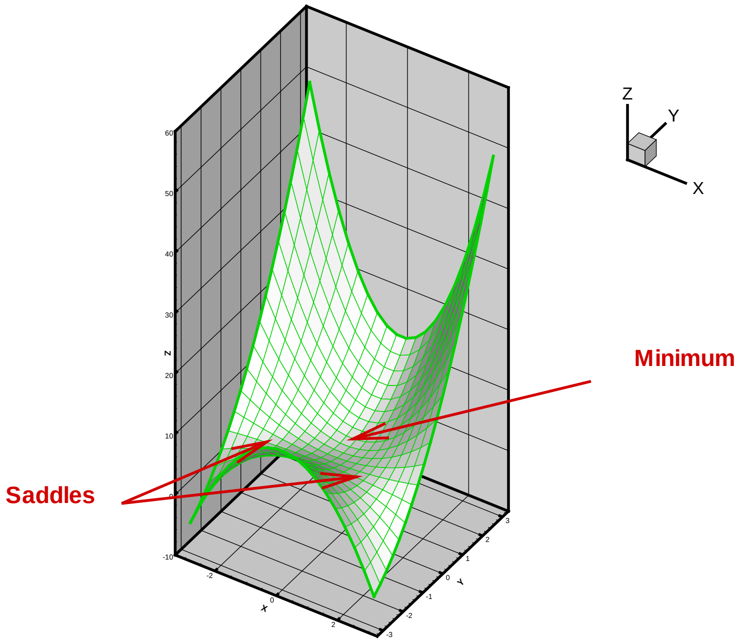

Figure 3: Plot of the function and its three stationary points.

3. **Taylor series**

Recall that the Taylor series of a function of a function of two variables, $f(x, y)$, about a point (x_0, y_0) is given by

$$
\begin{aligned}
f(x_0 + \epsilon, y_0 + \delta) &= f(x_0, y_0) \; + \\
&+ \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \epsilon \; + \; \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \delta \; + \\
&+ \frac{1}{2!}\left[\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)} \epsilon^2 \; + \; 2 \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x_0,y_0)} \epsilon\, \delta \; + \; \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)} \delta^2 \right] \\
&+ \cdots
\end{aligned}
$$

for "small" values of ϵ and δ.

We have already computed the required partial derivatives in question 1b. Evaluating them at (x_0, y_0) yields the required result.