2 Functions of multiple [two] variables

In many applications in science and engineering, a function of interest depends on multiple variables. For instance, the ideal gas law $p = \rho RT$ states that the pressure p is a function of both its density, ρ and its temperature, T. (The gas constant R is a material property and not a variable).

We will now show how to extend the analysis of functions of a single variable to functions of multiple variables. We will restrict ourselves to the case of two variables, i.e. functions of the form

$$z = z(x, y),$$

the extension to larger numbers of variables being relatively straightforward, apart from the fact that functions of three and more variables are somewhat harder to visualise...

2.1 Examples

Here are some plots of functions of two variables:

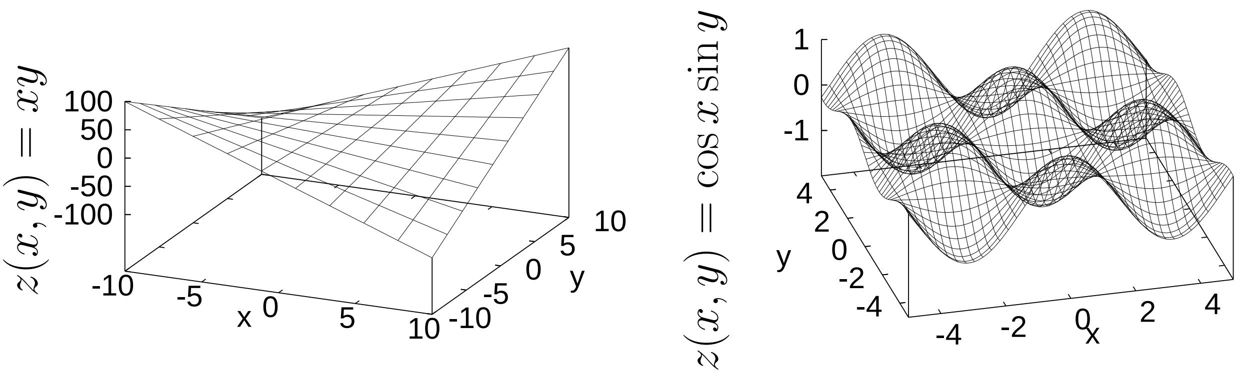


Figure 3: Functions of two variables: $z(x, y) = xy$ and $z(x) = \cos x \sin y$.

2.2 Partial derivatives

Functions of multiple variables can be differentiated with respect to either of their variables, the other variable being understood to be held constant during the differentiation. Such derivatives are known as *partial derivatives* and are distinguished from *ordinary derivatives* by using a ∂ instead of a d.

First derivatives: .

- For a function of two variables there are two partial derivatives
$$\frac{\partial z}{\partial x} = z_x$$
and
$$\frac{\partial z}{\partial y} = z_y.$$
- In general, the first derivatives are functions of x and y too.

Second derivatives: .

- For a function of two variables there are three second partial derivatives, defined as
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = z_{xx}$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = z_{yy}$$
and the mixed derivative
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = z_{xy}$$
where we usually[1] have
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$
i.e.
$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

Higher derivatives: .

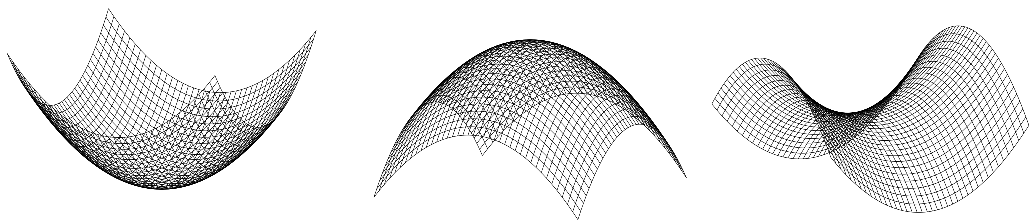
- Higher derivatives are again defined recursively, e.g.
$$\frac{\partial^5 z}{\partial x^3 \partial y^2} = \frac{\partial^3}{\partial x^3}\left(\frac{\partial^2 z}{\partial y^2}\right)$$
etc.

[1]Mathematicians obviously love to construct esoteric functions for which this property doesn't hold but it's rare to come across these in "real life". Strictly speaking, you can only exchange the order of the differentiation if the function $z(x, y)$ is sufficiently smooth. Most functions are.

2.3 Stationary points: Maxima and minima and saddles

Types of stationary points: .

- Functions of two variables can have stationary points of different types:



(a) A local minimum (b) A local maximum (c) A saddle point

Figure 4: Generic stationary points for a function of two variables.

Condition for a stationary point: .

- The function $z(x,y)$ has a "stationary point" at (x_0, y_0) if

$$\left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} = 0 \quad \underline{\textbf{and}} \quad \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)} = 0.$$

- This condition provides two equations for the two unknowns x_0 and y_0. These equations can have
  - No solution, in which case the function $z(x,y)$ has no stationary points.
  - A unique solution, in which case the function $z(x,y)$ has a single stationary point.
  - Multiple solutions, in which case the function $z(x,y)$ has multiple stationary points.

Classification of stationary points: .

- The nature of a stationary point is determined by the function's second derivatives. Here is a recipe for the classification of stationary points.

  For each stationary point (x_0, y_0):

  1. Determine the three second partial derivatives and evaluate them at the stationary point:
  $$A = \left.\frac{\partial^2 z}{\partial x^2}\right|_{(x_0,y_0)}, \quad B = \left.\frac{\partial^2 z}{\partial y^2}\right|_{(x_0,y_0)}, \quad \text{and} \quad C = \left.\frac{\partial^2 z}{\partial x \partial y}\right|_{(x_0,y_0)}.$$
  2. Compute the "discriminant"
  $$D = AB - C^2.$$
  3. Classify the stationary point according to the following cases:
  $$\begin{array}{lcl} D < 0 & \Longrightarrow & \text{Saddle point} \\ D > 0 \text{ and } \partial^2 z/\partial x^2 > 0 \ \left[\text{or } \partial^2 z/\partial y^2 > 0\right] & \Longrightarrow & \text{Local minimum} \\ D > 0 \text{ and } \partial^2 z/\partial x^2 < 0 \ \left[\text{or } \partial^2 z/\partial y^2 < 0\right] & \Longrightarrow & \text{Local maximum} \\ D = 0 & \Longrightarrow & \text{Test is inconclusive} \end{array}$$

2.4 Taylor series:

2.4.1 The leading-order terms

- The Taylor series of a function $z(x, y)$ about a point (x_0, y_0) provides an approximation of the function in the neighbourhood of (x_0, y_0):

$$\begin{aligned} z(x_0+\epsilon, y_0+\delta) &= z(x_0, y_0) + \\ &+ \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} \epsilon + \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)} \delta + \\ &+ \frac{1}{2!}\left[\left.\frac{\partial^2 z}{\partial x^2}\right|_{(x_0,y_0)} \epsilon^2 + 2\left.\frac{\partial^2 z}{\partial x \partial y}\right|_{(x_0,y_0)} \epsilon\,\delta + \left.\frac{\partial^2 z}{\partial y^2}\right|_{(x_0,y_0)} \delta^2 \right] + \cdots \end{aligned}$$

for "small" values of ϵ and δ.

As in the 1D case, this may also be written as

$$\begin{aligned} z(x, y) &= z(x_0, y_0) + \\ &+ \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} (x-x_0) + \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)} (y-y_0) + \\ &+ \frac{1}{2!}\left[\left.\frac{\partial^2 z}{\partial x^2}\right|_{(x_0,y_0)} (x-x_0)^2 + 2\left.\frac{\partial^2 z}{\partial x \partial y}\right|_{(x_0,y_0)} (x-x_0)\,(y-y_0) + \left.\frac{\partial^2 z}{\partial y^2}\right|_{(x_0,y_0)} (y-y_0)^2 \right] + \\ &+ \cdots \end{aligned}$$

for "small" values of $|x - x_0|$ and $|y - y_0|$.

2.4.2 The general form of the 2D Taylor series

The general expression for the Taylor series in two variables may be written as

$$f(x-x_0, y-y_0) = \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \left.\frac{\partial^n f}{\partial x^{n-k} \partial y^k}\right|_{(x_0,y_0)} (x-x_0)^{n-k} (y-y_0)^k \right\},$$

where

$$\binom{n}{k} = \frac{n!}{(n-k)!\;k!}$$

are the binomial coefficients. Recall that the n binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ may be obtained from the n-th row of Pascal's triangle:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

3 Example

We shall illustrate the various techniques by considering the function

$$z(x, y) = x^3 + 3y - y^3 - 3x.$$

Here is a sketch of the function:

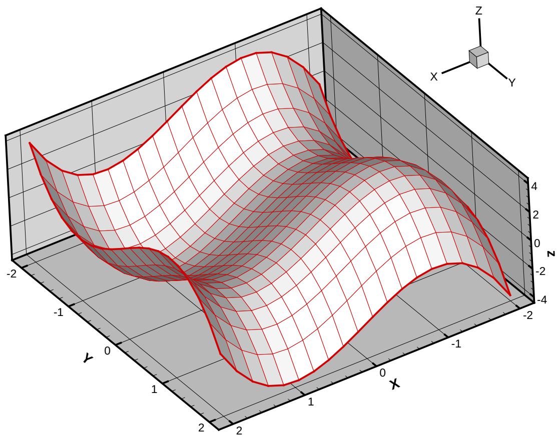


Figure 5: Sketch of the function $z(x, y) = x^3 + 3y - y^3 - 3x$.

Partial derivatives: .

- The partial derivatives are:

$$\frac{\partial z}{\partial x} = 3x^2 - 3.$$

$$\frac{\partial z}{\partial y} = 3 - 3y^2.$$

$$\frac{\partial^2 z}{\partial x^2} = 6x.$$

$$\frac{\partial^2 z}{\partial y^2} = -6y.$$

$$\frac{\partial^2 z}{\partial x \partial y} = 0.$$

Stationary points: .

- The coordinates (x_0, y_0) of any stationary points are given by the solution of the two equations:
$$\left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} = 3x_0^2 - 3 = 0$$
$$\left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)} = 3 - 3y_0^2 = 0.$$
[In the present example, these can be solved directly for $x_0 = \pm 1$ and $y_0 = \pm 1$, as the first equation only contains x_0 and the second one only y_0. In general, both equations will contain both unknowns and then have to be solved simultaneously.]
We therefore have four critical points:
$$\mathbf{P}_1 = (1,1) \quad \mathbf{P}_2 = (1,-1) \quad \mathbf{P}_3 = (-1,1) \quad \mathbf{P}_4 = (-1,-1).$$

- To classify the stationary points we evaluate the second derivatives in the following table:

Point	$A = 6x_0$	$B = -6y_0$	$C = 0$	$D = AB - C^2$	Classification
$\mathbf{P}_1 = (1,1)$	6	-6	0	-36	Saddle
$\mathbf{P}_2 = (1,-1)$	6	6	0	36	Local minimum
$\mathbf{P}_3 = (-1,1)$	-6	-6	0	36	Local maximum
$\mathbf{P}_4 = (-1,-1)$	-6	6	0	-36	Saddle

You should confirm these results by inspecting the plot shown at the beginning of the examples.

Taylor Expansion: .

- Finally, we determine the Taylor expansion of $z(x,y)$ about the point $(x,y) = (2,1)$, where $z(2,1) = 4$. We start by evaluating the derivatives:
$$\left.\frac{\partial z}{\partial x}\right|_{(2,1)} = 9$$
$$\left.\frac{\partial z}{\partial y}\right|_{(2,1)} = 0$$
$$\left.\frac{\partial^2 z}{\partial x^2}\right|_{(2,1)} = 12$$
$$\left.\frac{\partial^2 z}{\partial y^2}\right|_{(2,1)} = -6$$
$$\left.\frac{\partial^2 z}{\partial x \partial y}\right|_{(2,1)} = 0.$$

- In the vicinity of the point $(x, y) = (2, 1)$, i.e. for small values of ϵ and δ, the function $z(x, y)$ can therefore be approximated as

$$z(x = 2+\epsilon, y = 1+\delta) = \underbrace{4}_{z(2,1)} + \underbrace{9}_{\frac{\partial z}{\partial x}\big|_{(2,1)}} \epsilon + \underbrace{0}_{\frac{\partial z}{\partial y}\big|_{(2,1)}} \delta + \\ + \frac{1}{2!}\Bigg[\underbrace{12}_{\frac{\partial^2 z}{\partial x^2}\big|_{(2,1)}} \epsilon^2 + 2 \times \underbrace{0}_{\frac{\partial^2 z}{\partial x \partial y}\big|_{(2,1)}} \delta\,\epsilon + \underbrace{(-6)}_{\frac{\partial^2 z}{\partial y^2}\big|_{(2,1)}} \delta^2 \Bigg] + ...$$

So

$$y(x = 2+\epsilon, y = 1+\delta) = 4 + 9\epsilon + 6\epsilon^2 - 3\delta^2 + ...$$

or

$$y(x, y) = 4 + 9(x-2) + 6(x-2)^2 - 3(y-1)^2 + ...$$