Lecture Notes for 2M1 – Q-Stream

Dr Matthias Heil

School of Mathematics
University of Manchester

M.Heil@maths.man.ac.uk

Course webpage:

http://www.maths.man.ac.uk/~mheil/Lectures/2M1

Note:

- This part of the course, dealing with functions of two variables and partial differential equations (PDEs), is taught during weeks 1-5.
- The course web page provides online access to the lecture notes, example sheets and other handouts and announcements.
- Most of the material will be taught in "chalk and talk" mode. If OHP transparencies are used, copies will be made available (after the lecture) on this page.
- Please consult the service course page

  http://www.maths.man.ac.uk/service

  for details on how to get hold of material for the other parts of the course.
- Please note that the lecture notes only summarise the main results and will generally be handed out *after* the material has been covered in the lecture. You are expected take notes during the classes.

1 Reminder: Functions of a single variable

A function $y = y(x)$ is a function of a single variable.

Examples

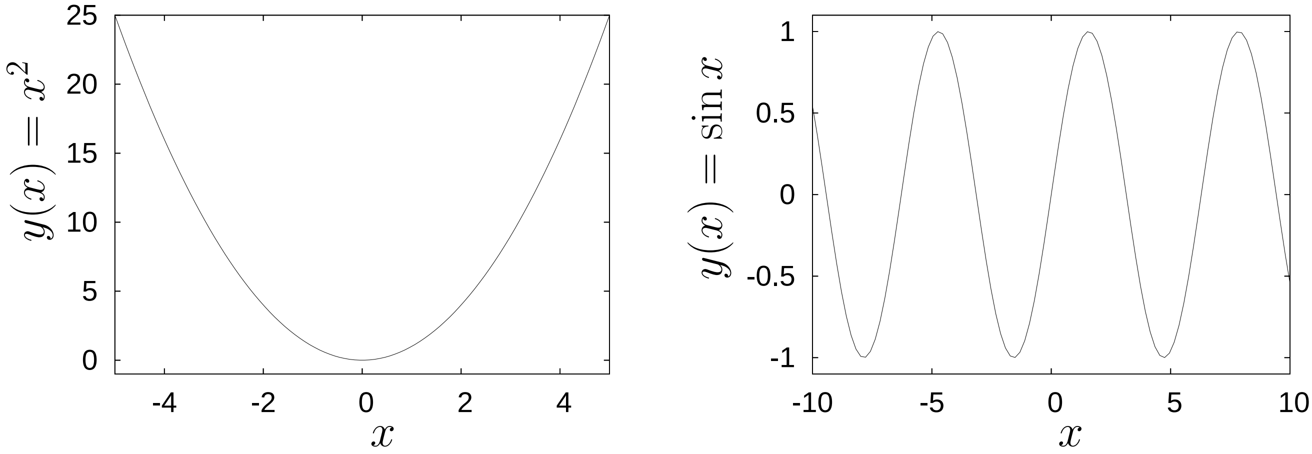


Figure 1: Functions of a single variable: $y(x) = x^2$ and $y(x) = \sin x$.

1.1 [Ordinary] derivatives

First derivative:

$$y' = \frac{dy}{dx}$$

- The first derivative represents the slope of the curve $y = y(x)$.
- In general, y' is a function of x too.

Second derivative:

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

- The second derivative is the derivative of the first derivative.
- The second derivative indicates the curvature of the curve $y = y(x)$.

Higher derivatives:

$$y^{(n)} = \frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\cdots\left(\frac{dy}{dx}\right)\cdots\right)\right)\right)$$

- Higher derivatives are defined recursively: The n-th derivative is the derivative of the $n-1$-th derivative.

1.2 Stationary points: Maxima and minima

Condition for a stationary point: .

- The function $y(x)$ has a "stationary point" at x_0 if

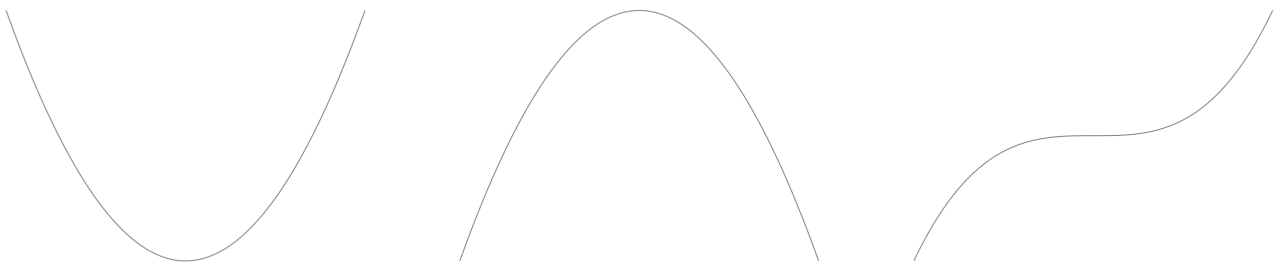
$$\left.\frac{dy}{dx}\right|_{x_0} = 0,$$

where the $(...)|_{x_0}$ notation indicates that the expression in the round brackets is to be evaluated at $x = x_0$.

Classification of stationary points: .

- The nature of a stationary point is determined by the function's second derivative:

$$\left.\frac{d^2y}{dx^2}\right|_{x_0} \begin{cases} > 0 & \Longrightarrow \quad \text{Local minimum} \\ < 0 & \Longrightarrow \quad \text{Local maximum} \\ = 0 & \Longrightarrow \quad \text{Test is not conclusive (curve too flat; e.g. at an inflection point.)} \end{cases}$$



(a) A local minimum (b) A local maximum (c) An inflection point

Figure 2: Generic stationary points for a function of one variable.

1.3 Taylor series:

- The Taylor series of a function $y(x)$ about a point $x = x_0$ provides an approximation of the function in the neighbourhood of x_0:

$$y(x_0 + \epsilon) = y(x_0) + \left.\frac{dy}{dx}\right|_{x_0} \epsilon + \frac{1}{2!} \left.\frac{d^2y}{dx^2}\right|_{x_0} \epsilon^2 + \frac{1}{3!} \left.\frac{d^3y}{dx^3}\right|_{x_0} \epsilon^3 + ...$$

for "small" $|\epsilon|$.

Here $n! = 1 \times 2 \times 3 \times .... \times (n-1) \times n$ is the factorial.

The Taylor expansion may also be written as

$$y(x) = y(x_0) + \left.\frac{dy}{dx}\right|_{x_0} (x - x_0) + \frac{1}{2!} \left.\frac{d^2y}{dx^2}\right|_{x_0} (x - x_0)^2 + \frac{1}{3!} \left.\frac{d^3y}{dx^3}\right|_{x_0} (x - x_0)^3 + ...$$

for "small" values of $|x - x_0|$.