2M1 – Q-STREAM: SOLUTIONS [1] II

1. **Solution of PDEs "by inspection"**

   (a) To verify that the function $u(x, y) = x - y$ is a solution of the PDE
   $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$$
   we form the required partial derivatives
   $$\frac{\partial u}{\partial x} = 1$$
   and
   $$\frac{\partial u}{\partial y} = -1,$$
   showing that their sum is
   $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0,$$
   as required.

   (b) To show that $u(x, t) = \sin(x + t) + \cos(x - t)$ is a solution of the 1D linear wave equation
   $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$
   we form the required partial derivatives
   $$\frac{\partial u}{\partial x} = \cos(x + t) - \sin(x - t)$$
   (remember the chain rule!)
   $$\frac{\partial^2 u}{\partial x^2} = -\sin(x + t) - \cos(x - t)$$
   and
   $$\frac{\partial u}{\partial t} = \cos(x + t) + \sin(x - t)$$

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$$\frac{\partial^2 u}{\partial t^2} = -\sin(x+t) - \cos(x-t),$$

showing that the two second partial derivatives are identical, as required.

(c) We determine the required derivatives of $u(x,t) = e^{at}(\sin x - bx^2)$:

$$\frac{\partial u}{\partial t} = a\ e^{at}(\sin x - bx^2)$$

and

$$\frac{\partial^2 u}{\partial x^2} = e^{at}(-\sin x - 2b).$$

Inserting them into the 1D unsteady heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

yields

$$a\ e^{at}(\sin x - bx^2) = e^{at}(-\sin x - 2b).$$

This can be rewritten as

$$e^{at}\left[\sin x\ (a+1) - bx^2 + 2b\right] = 0.$$

Since $e^{at} \neq 0$, the expression in the square brackets has to vanish for *all* values of the independent variable, x. This is only possible if the coefficients multiplying the various (linearly independent) functions vanish. This requires $a = -1$ and $b = 0$.

2. **Separation of variables for the 1D linear wave equation**

We follow the procedure discussed in the lecture:

Step 1: Write the unknown function of two variables as a product of two functions of a single variable:

$$u(x,t) = X(x)\ T(t).$$

Step 2: Insert this "ansatz" into the PDE and differentiate.

$$X(x)\ \ddot{T}(t) = X''(x)\ T(t).$$

Step 3: Separate the variables, i.e. move all functions that only depend on t onto one side of the equation and all functions that depend only on x onto the other one:

$$\frac{\ddot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

Since the LHS now only depends on t and the RHS only on x, both must, in fact, be constant and we arbitrarily call the (as yet unknown) constant $-\omega^2$ to obtain

$$\frac{\ddot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = const. = -\omega^2$$

Step 4: Solve the "spatial" equation for $X(x)$

$$X''(x) + \omega^2 \; X(x) = 0 \quad \Longrightarrow \quad X(x) = \widehat{A}\sin(\omega x) + \widehat{B}\cos(\omega x)$$

for some constants $\widehat{A}$ and $\widehat{B}$.

Step 5: Apply the boundary conditions:

$$u(x = 0, t) = X(0) \; T(t) = 0 \quad \Longrightarrow \quad X(0) = 0 \quad \Longrightarrow \quad \widehat{B} = 0.$$

$$u(x = 1, t) = X(1)\, T(t) = 0 \quad \Longrightarrow \quad X(1) = 0 \quad \Longrightarrow \quad \widehat{A}\sin(\omega) = 0.$$

The latter equation can be satisfied either by setting $\widehat{A} = 0$ or $\omega = 0$ (in which case $u(x, t) \equiv 0$ which cannot satisfy the initial conditions) or by setting

$$\omega = \pi, \; 2\pi, \; 3\pi, ...$$

while leaving $\widehat{A}$ undetermined.

Step 6: Solve the "temporal equation" for $T(t)$:

$$\ddot{T}(t) + \omega^2 \; T(t) = 0 \quad \Longrightarrow \quad T(t) = \widehat{C}\sin(\omega t) + \widehat{D}\cos(\omega t)$$

for some constants $\widehat{C}$ and $\widehat{D}$.

Step 7: Combine the spatial and temporal factors and combine any superfluous undetermined constants:

$$u(x, t) = \widehat{A}\sin(\omega x)\big(\widehat{C}\sin(\omega t) + \widehat{D}\cos(\omega t)\big) = \sin(\omega x)\big(A\sin(\omega t) + B\cos(\omega t)\big)$$

where $A = \widehat{A}\widehat{C}$ and $B = \widehat{A}\widehat{D}$.

Step 8: Apply the initial conditions

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin(3\pi x) = A\omega\sin(\omega x) \quad \Longrightarrow \quad \omega = 3\pi \;\text{ and }\; A\omega = 1, \text{ i.e. } A = 1/(3\pi).$$

$$u(x, t = 0) = 0 = B\sin(\omega x) \quad \Longrightarrow \quad B = 0.$$

Step 9: Done! The solution is

$$u(x,t) = \frac{1}{3\pi}\sin(3\pi t)\sin(3\pi x).$$

3. **Separation of variables for the 1D unsteady heat equation**

Recall that this problem may be interpreted as describing the spatial and temporal evolution of the temperature in a thin, well-insulated metal bar whose ends are held at zero tempature. Physically we expect the initial tempature distribution, $u_0(x) = \sin(x)$, to decay towards a state in which the temparature is zero everywhere.

The separation of variables method follow the same steps as in the linear wave example – the main difference being that we only have a first temporal derivative.

Step 1: Write the unknown function of two variables as a product of two functions of a single variable:

$$u(x,t) = X(x)\;T(t).$$

Step 2: Insert this "ansatz" into the PDE and differentiate.

$$X(x)\;\dot{T}(t) = X''(x)\;T(t).$$

Step 3: Separate the variables, i.e. move all functions that only depend on t onto one side of the equation and all functions that depend only on x onto the other one:

$$\frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

Since the LHS now only depends on t and the RHS only on x, both must, in fact, be constant and we arbitrarily call the (as yet unknown) constant $-\omega^2$ to obtain

$$\frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = const. = -\omega^2$$

Step 4: Solve the "spatial" equation for $X(x)$

$$X''(x) + \omega^2\, X(x) = 0 \quad \Longrightarrow \quad X(x) = \widehat{A}\sin(\omega x) + \widehat{B}\cos(\omega x)$$

for some constants $\widehat{A}$ and $\widehat{B}$.

Step 5: Apply the boundary conditions:

$$u(x=0,t) = X(0)\,T(t) = 0 \quad \Longrightarrow \quad X(0) = 0 \quad \Longrightarrow \quad \widehat{B} = 0.$$

$$u(x=\pi,t) = X(\pi)\,T(t) = 0 \quad \Longrightarrow \quad X(\pi) = 0 \quad \Longrightarrow \quad \widehat{A}\sin(\omega\pi) = 0.$$

The latter equation can be satisfied either by setting $\widehat{A} = 0$ or $\omega = 0$ (in which case $u(x,t) \equiv 0$ which cannot satisfy the initial conditions) or by setting

$$\omega = 1,\ 2,\ 3, ...$$

while leaving $\widehat{A}$ undetermined.

Step 6: Solve the "temporal equation" for $T(t)$:

$$\dot{T}(t) + \omega^2\, T(t) = 0 \quad \Longrightarrow \quad T(t) = \widehat{C}\exp(-\omega^2 t).$$

for some constant $\widehat{C}$. Note that if we had chosen another sign for the separation constant, the tempature in the metal bar would increase exponentially – not what we would expect!

Step 7: Combine the spatial and temporal factors and combine any superfluous undetermined constants:

$$u(x,t) = \widehat{A}\sin(\omega x)\ \widehat{C}\exp(-\omega^2 t) = A\ \sin(\omega x)\exp(-\omega^2 t)$$

where $A = \widehat{A}\widehat{C}$.

Step 8: Apply the initial conditions

$$u(x,t=0) = \sin(x) = A\sin(\omega x) \quad \Longrightarrow \quad A = 1 \quad \text{and} \quad \omega = 1$$

Step 9: Done! The solution is

$$u(x,t) = \exp(-t)\sin(x).$$

Compare to the result of question 1c.