Numerical Linear Algebra Applications of Tropical Mathematics

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Prospects in Mathematics
December 18–19, 2012
Manchester

A Definition of Numerical Linear Algebra

Numerical linear algebra is the study of algorithms for performing linear algebra computations.

For example,

- Solve a system of linear equations, $Ax = b$, $A \in \mathbb{R}^{n\times n}$.
- Find eigenvalues and eigenvectors, $Ax = \lambda x$, $A \in \mathbb{R}^{n\times n}$.
- Compute (when it exists) e^A, $A^{1/2}$, $\log(A)$, $A \in \mathbb{R}^{n\times n}$.
- Find $x \in \mathbb{R}^n$ minimizing $\|Ax - b\|$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ $(m \geq n)$.

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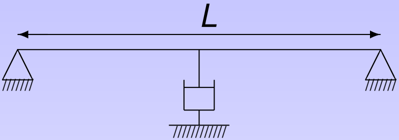
Also: what if we don't know whether A is nonsingular?

Beam Problem



- Transverse displacement $u(x, t)$ governed by

$$\rho A \frac{\partial^2 u}{\partial t^2} + c(x)\frac{\partial u}{\partial t} + EI\frac{\partial^4 u}{\partial x^4} = 0.$$

$$u(0, t) = u''(0, t) = u(L, t) = u''(L, t) = 0.$$

- Separation of variables $u(x, t) = e^{\lambda t} v(x, \lambda)$ yields the **eigenvalue problem for the free vibrations**:

$$\lambda^2 \rho A v(x, \lambda) + \lambda c(x) v(x, \lambda) + EI\frac{\partial^4}{\partial x^4} v(x, \lambda) = 0.$$

Discretized Beam Problem

Finite element method leads to

$$Q(\lambda)v = (\lambda^2 M + \lambda D + K)v = 0 \qquad (*)$$

with symmetric $M, D, K \in \mathbb{R}^{n \times n}$.

- $(*)$ is a **quadratic eigenvalue problem** (generalizes $Av = \lambda v$).
- λ is an eigenvalue with corresponding eigenvector v.
- $Q(\lambda)$ has $2n$ eigenvalues, solutions of $\det(Q(\lambda)) = 0$.

Solution Process

Find all λ and v satisfying $Q(\lambda)v = (\lambda^2 M + \lambda D + K)v = 0$.

- Commonly solved by linearization:
  - Convert $Q(\lambda)v = 0$ into $(\mathcal{A} - \lambda\mathcal{B})\xi = 0$, e.g.,

$$\mathcal{A} - \lambda\mathcal{B} = \begin{bmatrix} K & 0 \\ 0 & I \end{bmatrix} - \lambda \begin{bmatrix} -D & -M \\ I & 0 \end{bmatrix}, \quad \xi = \begin{bmatrix} v \\ \lambda v \end{bmatrix}.$$

  - Solve $(\mathcal{A} - \lambda\mathcal{B})\xi = 0$ with a numerical method (e.g., QZ algorithm).
  - Recover eigenvectors of $Q(\lambda)$ from those of $\mathcal{A} - \lambda\mathcal{B}$.

Eigenvalues of $Q(\lambda) = \lambda^2 M + \lambda D + K$

When M, K are nonsingular then theoretically

$$C_1(\lambda) = \lambda \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} D & K \\ -I & 0 \end{bmatrix},$$

$$L_1(\lambda) = \lambda \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix} + \begin{bmatrix} D & K \\ K & 0 \end{bmatrix},$$

$$L_2(\lambda) = \lambda \begin{bmatrix} 0 & M \\ M & D \end{bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}$$

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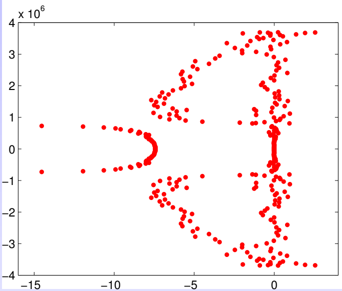
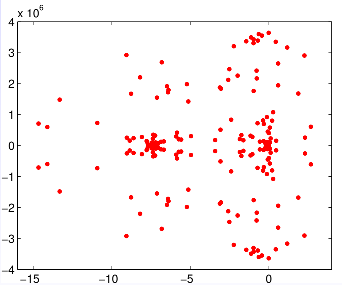
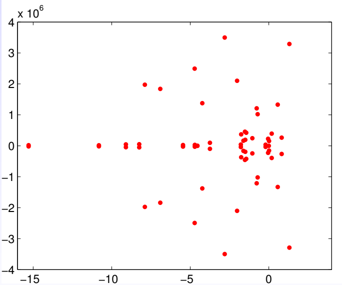
What about numerically? Let's try for the beam problem.

```
eC1 = eig([D K; -I O],-[M O; O I]); % C1
eL1 = eig([D K; K O],-[M O; O -K]); % L1
eL2 = eig([-M O; O K],-[O M; M D]); % L2
plot(eC1,'.r');plot(eL1,'.r');plot(eL2,'.r')
```

Computed Spectra of C_1, L_1 and L_2







Conditioning and Backward Error

- **Condition number** measures sensitivity of the solution of a problem to perturbations in the data.
- **Backward error** measures how well the problem has been solved.

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- Can we modify the problem into an equivalent one whose solution is **less sensitive** to perturbations?
- Can we develop a **numerically stable** procedure to solve the problem?

Eigenvalue Parameter Scaling

Let $\lambda = \mu\gamma$, $\gamma \neq 0$ and convert $Q(\lambda) = \lambda^2 M + \lambda D + K$ to

$$Q(\mu\gamma) = \mu^2(\gamma^2 M) + \mu(\gamma D) + K = \mu^2 \widetilde{M} + \mu \widetilde{D} + \widetilde{K} =: \widetilde{Q}(\mu).$$

Can we choose γ such that

- the standard solution process in numerically stable,
- the eigenvalues of the linearizations are less sensitive to perturbations?

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Try $\gamma = \exp(r)$, where r is a **tropical root** of a **tropical scalar quadratic** (Gaubert & Sharify 2009).

Tropical Scalar Polynomials

- Let $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ be the **tropical semiring** with

$$a \oplus b = \max(a, b), \quad a \otimes b = a + b \quad \text{for all } a, b \in \mathbb{R} \cup \{-\infty\}.$$

- The piecewise affine function

$$p(x) = \bigoplus_{k=0}^{d} p_k \otimes x^{\otimes k} = \max_{0 \le k \le d} (p_k + kx), \quad p_k \in \mathbb{R} \cup \{-\infty\}$$

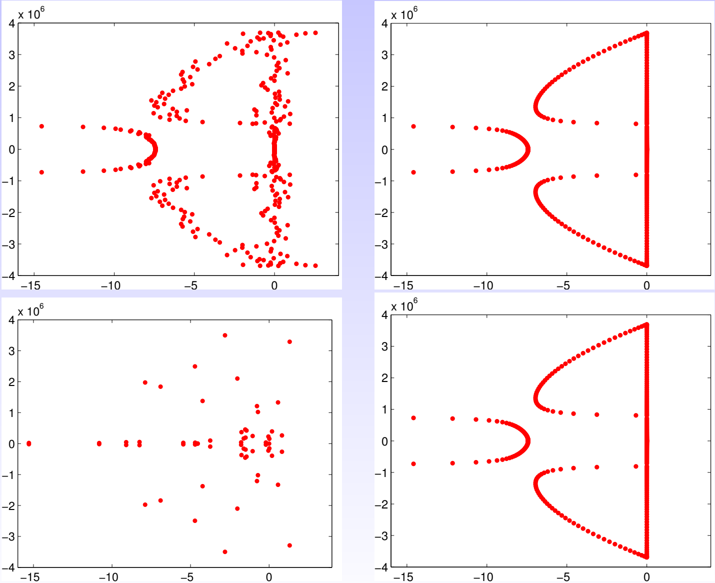
is a **tropical polynomial** of degree d.

- The **tropical roots** of $p(x)$ are the points of nondifferentiability of $p(x)$.

Tropical Roots

- $\max(1, -1 + x, 2x, -2 + 3x)$ has roots $1/2$, $1/2$ and 2.
- Tropical roots can be computed in **linear time**.
- **Classical roots** of $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ can be **bounded in terms of tropical roots** of $p_{\mathrm{trop}}(x) = \max(\log|a_0|, \log|a_1| + x, \ldots, \log|a_n| + nx)$.
- Let r_1, r_2 be the **tropical roots** of $p_{\mathrm{trop}}(r) = \max(\log(\|K\|), \log(\|D\|) + r, \log(\|M\|) + 2r)$. Under some assumptions, e^{r_1} **and** e^{r_2} are **good approximations of largest and smallest eigenvalues in modulus** of $Q(\lambda) = \lambda^2 M + \lambda D + K$.

Spectrum of C_1, L_2 before/after Scaling



Where to Study Tropical Mathematics?

Vibrant area of research in both pure and applied mathematics.

- **Birmingham**: Peter Butkovič.
- **Manchester**: Marianne Johnson, Mark Kambites, Mark Muldoon.
- **Warwick**: Diane Maclagan.

Where to Study Numerical Linear Algebra?

- **Bath**: Melina Freitag and Alastair Spence.
- **Manchester**: Jack Dongarra, Stefan Güttel, Nick Higham, Françoise Tisseur.
- **Oxford**: Nick Trefethen, Andy Wathen.
- **Strathclyde** Des Higham, Philip Knight, Alison Ramage.