Applications of Random Matrix Theory

Christopher Hughes

THE UNIVERSITY of York

University of Manchester
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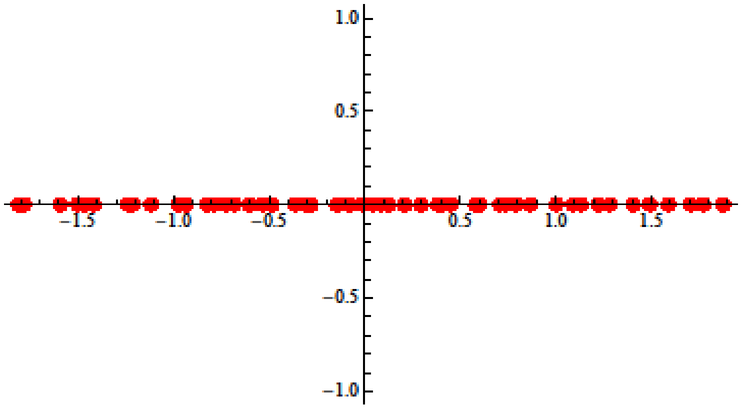
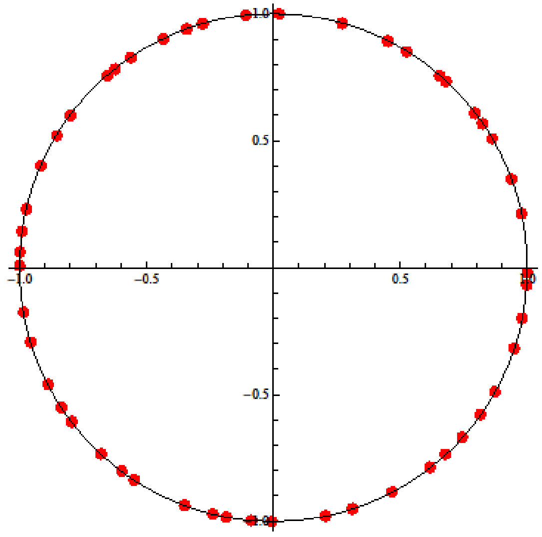
Matrices - Recall

- Eigenvalues $M\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$
- Symmetry important:
  - Real symmetric means $M = M^t$
  - Eigenvalues are real.



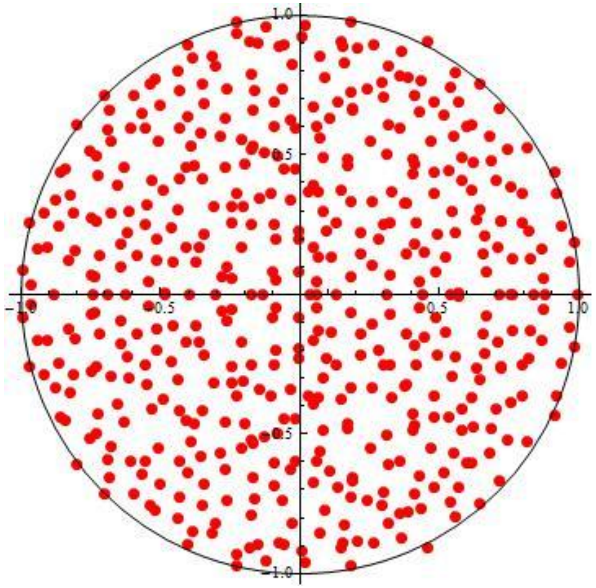
Matrices - Recall

- Eigenvalues $M\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$
- Symmetry important:
  - Unitary means $MM^{\dagger} = I$
  - eigenvalues lie on the unit circle.



Matrices - Recall

- Eigenvalues $M\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$
- Symmetry important:
  - No symmetry
  - eigenvalues lie in the complex plane.



Nuclear Resonance Spectroscopy – Wigner's insight

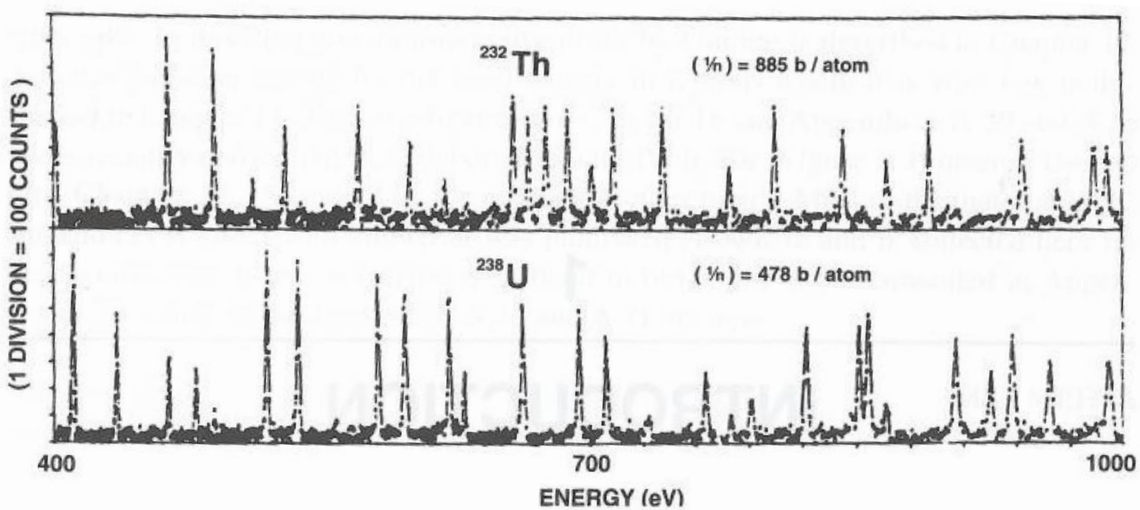


Figure 1.1. Slow neutron resonance cross-sections on thorium 232 and uranium 238 nuclei. Reprinted with permission from The American Physical Society, Rahn et al., Neutron resonance spectroscopy, X, *Phys. Rev. C* 6, 1854–1869 (1972).

Wigner's insight

- Quantum mechanics controls the energy levels, via Schrödinger's Equation.
- Too complicated!
- Keep the symmetry (as imposed by physics) but let everything else be random.

Data for Erbium (^{166}Er)

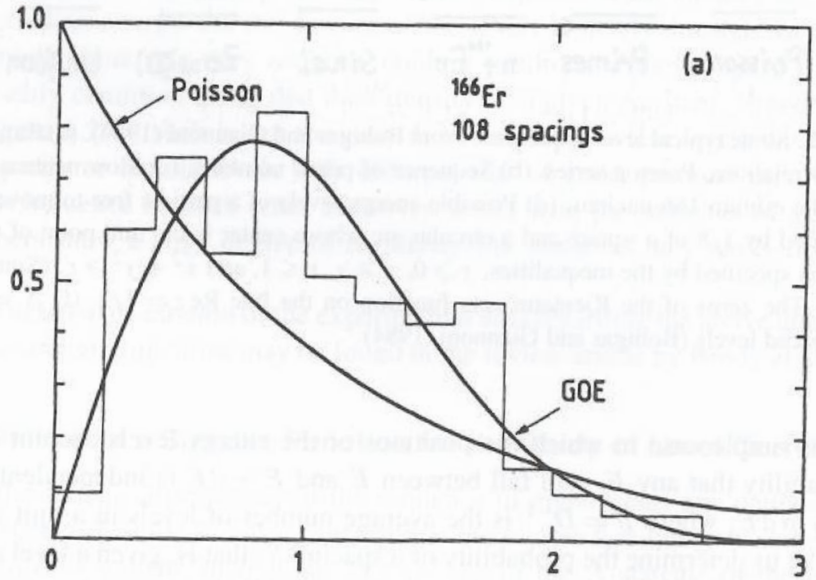


Figure 1.3. The probability density for the nearest neighbor spacings in slow neutron resonance levels of erbium 166 nucleus. The histogram shows the first 108 levels observed. The solid curves correspond to the Poisson distribution, i.e. no correlations at all, and that for the eigenvalues of a real symmetric random matrix taken from the Gaussian orthogonal ensemble (GOE). Reprinted with permission from The American Physical Society, Liou et al., Neutron resonance spectroscopy data, *Phys. Rev. C* 5 (1972) 974–1001.

Data for 27 different nuclei

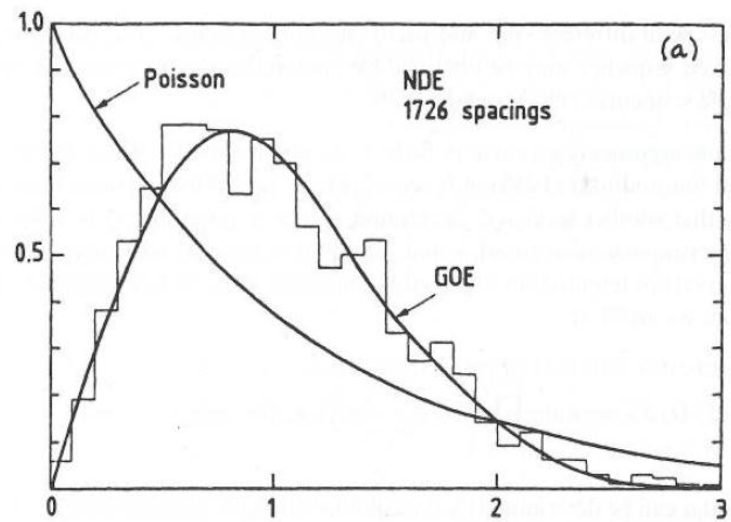
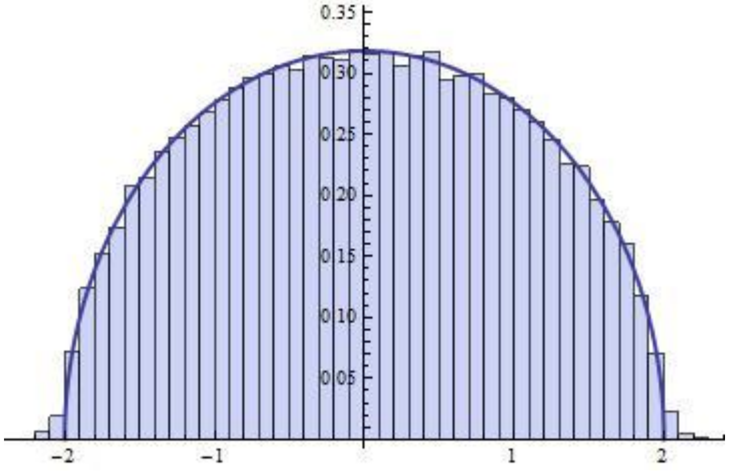


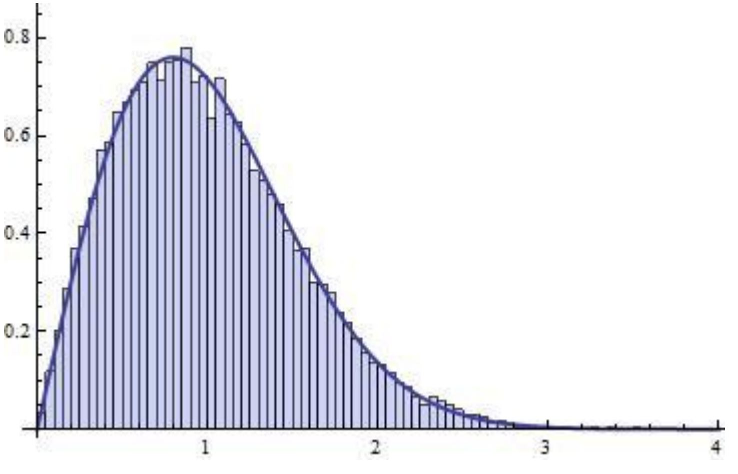
Figure 1.4. Level spacing histogram for a large set of nuclear levels, often referred to as nuclear data ensemble. The data considered consists of 1407 resonance levels belonging to 30 sequences of 27 different nuclei: (i) slow neutron resonances of Cd(110, 112, 114), Sm(152, 154), Gd(154, 156, 158, 160), Dy(160, 162, 164), Er(166, 168, 170), Yb(172, 174, 176), W(182, 184, 186), Th(232) and U(238); (1146 levels); (ii) proton resonances of Ca(44) ($J = 1/2+$), Ca(44) ($J = 1/2-$), and Ti(48) ($J = 1/2+$); (157 levels); and (iii) (n, γ)-reaction data on Hf(177) ($J = 3$), Hf(177) ($J = 4$), Hf(179) ($J = 4$), and Hf(179) ($J = 5$); (104 levels). The data chosen in each sequence is believed to be complete (no missing levels) and pure (the same angular momentum and parity). For each of the 30 sequences the average quantities (e.g. the mean spacing, spacing/mean spacing, number variance μ_2, etc., see Chapter 16) are computed separately and their aggregate is taken weighted according to the size of each sequence. The solid curves correspond to the Poisson distribution, i.e. no correlations at all, and that for the eigenvalues of a real symmetric random matrix taken from the Gaussian orthogonal ensemble (GOE). Reprinted with permission from Kluwer Academic Publishers, Bohigas O., Haq R.U. and Pandey A., Fluctuation properties of nuclear energy levels and widths, comparison of theory with experiment, in: *Nuclear Data for Science and Technology*, Bökhoff K.H. (Ed.), 809–814 (1983).

First simple questions

- Eigenvalue distribution



- Nearest neighbour spacing



Symmetries are important

- GOE:
  - $M = \frac{1}{\sqrt{2}}(A + A^t)$ where the entries of A are iid $N\left(0, \frac{1}{\sqrt{n}}\right)$.
  - M is a real symmetric matrix.
  - Measure invariant under orthogonal transforms.
- GUE:
  - $M = \frac{1}{\sqrt{2}}(A + A^\dagger)$ where the entries of A are iid complex normal $N\left(0, \frac{1}{2\sqrt{n}}\right) + iN\left(0, \frac{1}{2\sqrt{n}}\right)$.
  - M is a hermitian matrix
  - Measure invariant under unitary transforms.
- And GSE (and Dyson's COE, CUE, CSE) and many others.

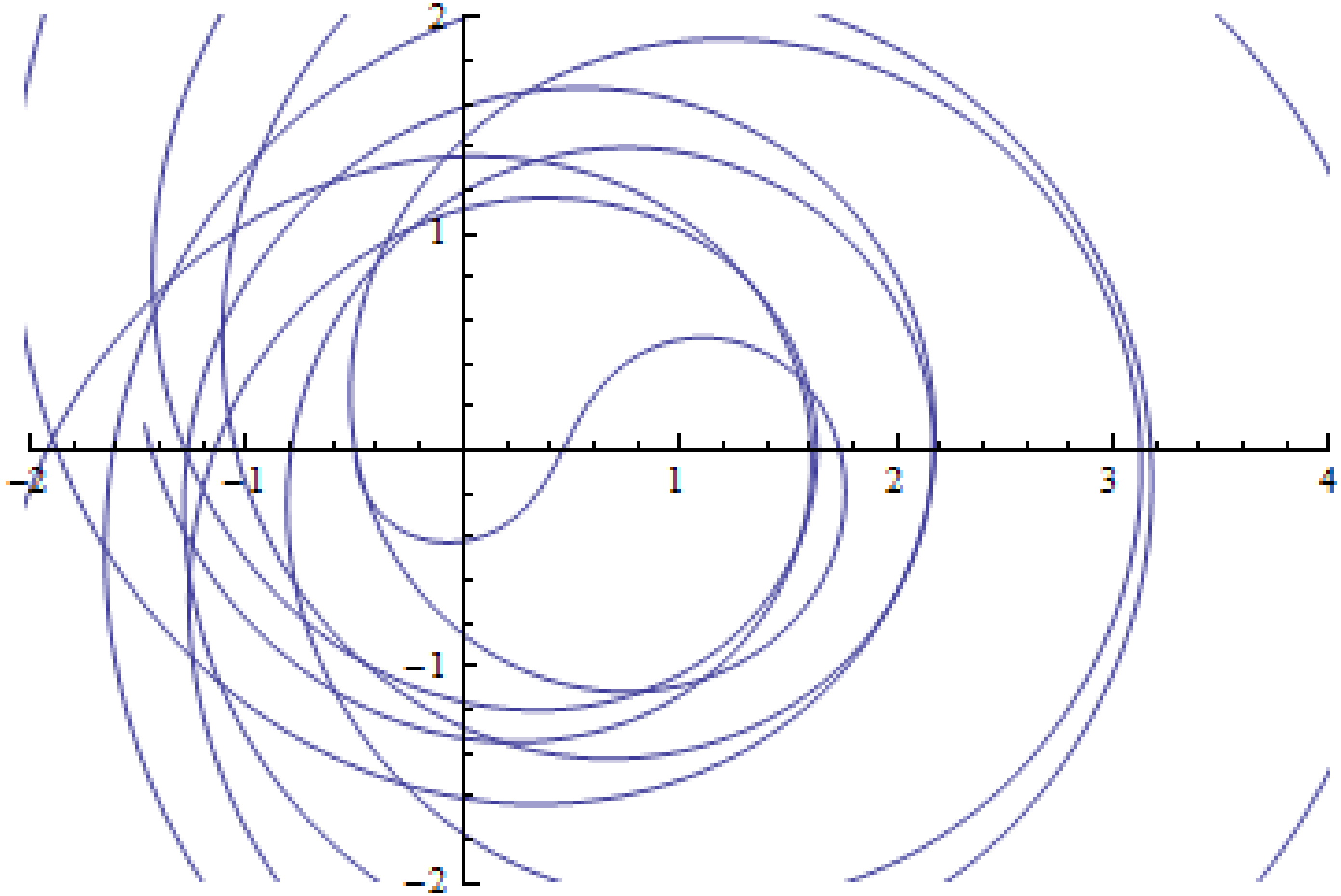
Maths connections & questions

- Free probability
- Riemann-Hilbert problem
- Orthogonal polynomials
- Brownian motion
- Painlevé equations
- Combinatorics
- Universality
- Distribution of largest eigenvalue
- …and many more…

Riemann zeta function

- The Riemann zeta function is of fundamental importance in number theory.
  - Understand the distribution of prime numbers.
  - Prototypical L-function (Dirichlet, elliptic curves, modular forms, etc).
  - An interesting and challenging function to understand in its own right.
  - Become a millionaire!

Riemann zeta function



Alan Turing and zeros of zeta

SOME CALCULATIONS OF THE RIEMANN ZETA-FUNCTION

By A. M. TURING

[Received 29 February 1952.—Read 20 March 1952]

Introduction

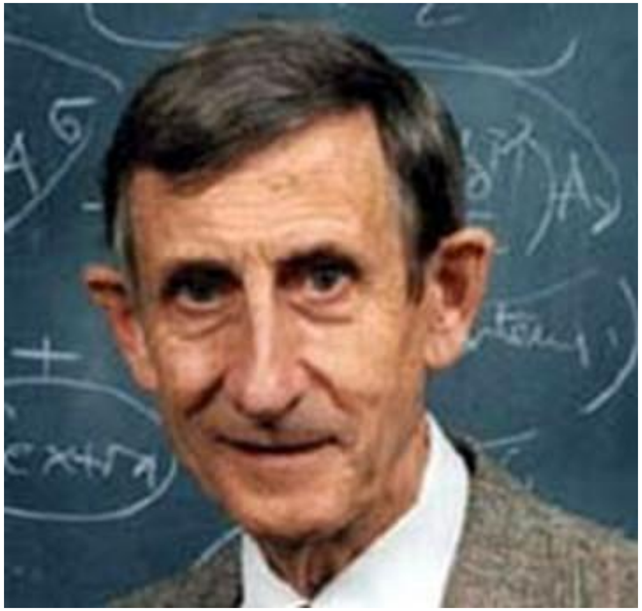
In June 1950 the Manchester University Mark 1 Electronic Computer was used to do some calculations concerned with the distribution of the zeros of the Riemann zeta-function. It was intended in fact to determine whether there are any zeros not on the critical line in certain particular intervals. The calculations had been planned some time in advance, but had in fact to be carried out in great haste. If it had not been for the fact that the computer remained in serviceable condition for an unusually long period from 3 p.m. one afternoon to 8 a.m. the following morning it is probable that the calculations would never have been done at all. As it was, the interval $2\pi . 63^2 < t < 2\pi . 64^2$ was investigated during that period, and very little more was accomplished.

The calculations were done in an optimistic hope that a zero would be found off the critical line, and the calculations were directed more towards

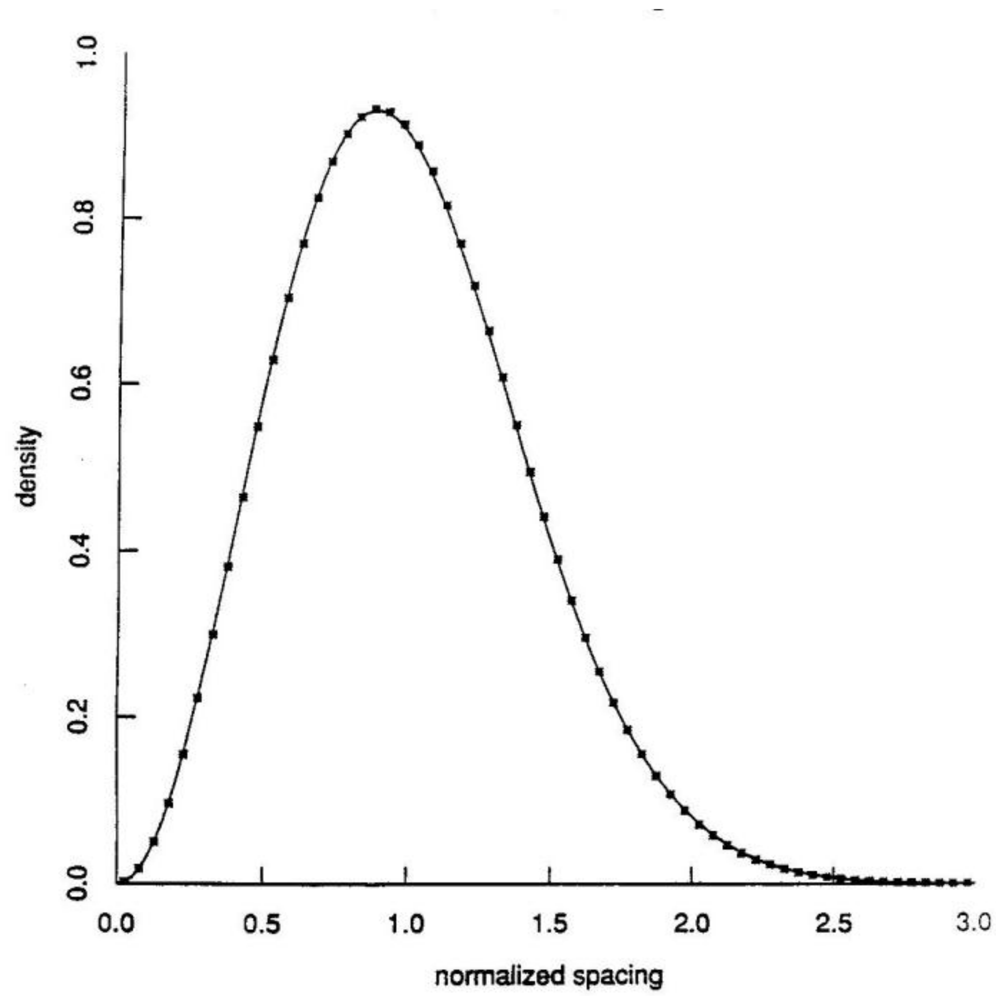
'There is more chance of him proving Riemann's hypothesis than wearing a sarong'

Wayne Rooney will face Portugal tonight as England's new sporting icon. But, says Andrew Anthony, it is more than just a physical genius that makes Rooney special - he's a bit of rough, an antidote to the glamorous Beckham years

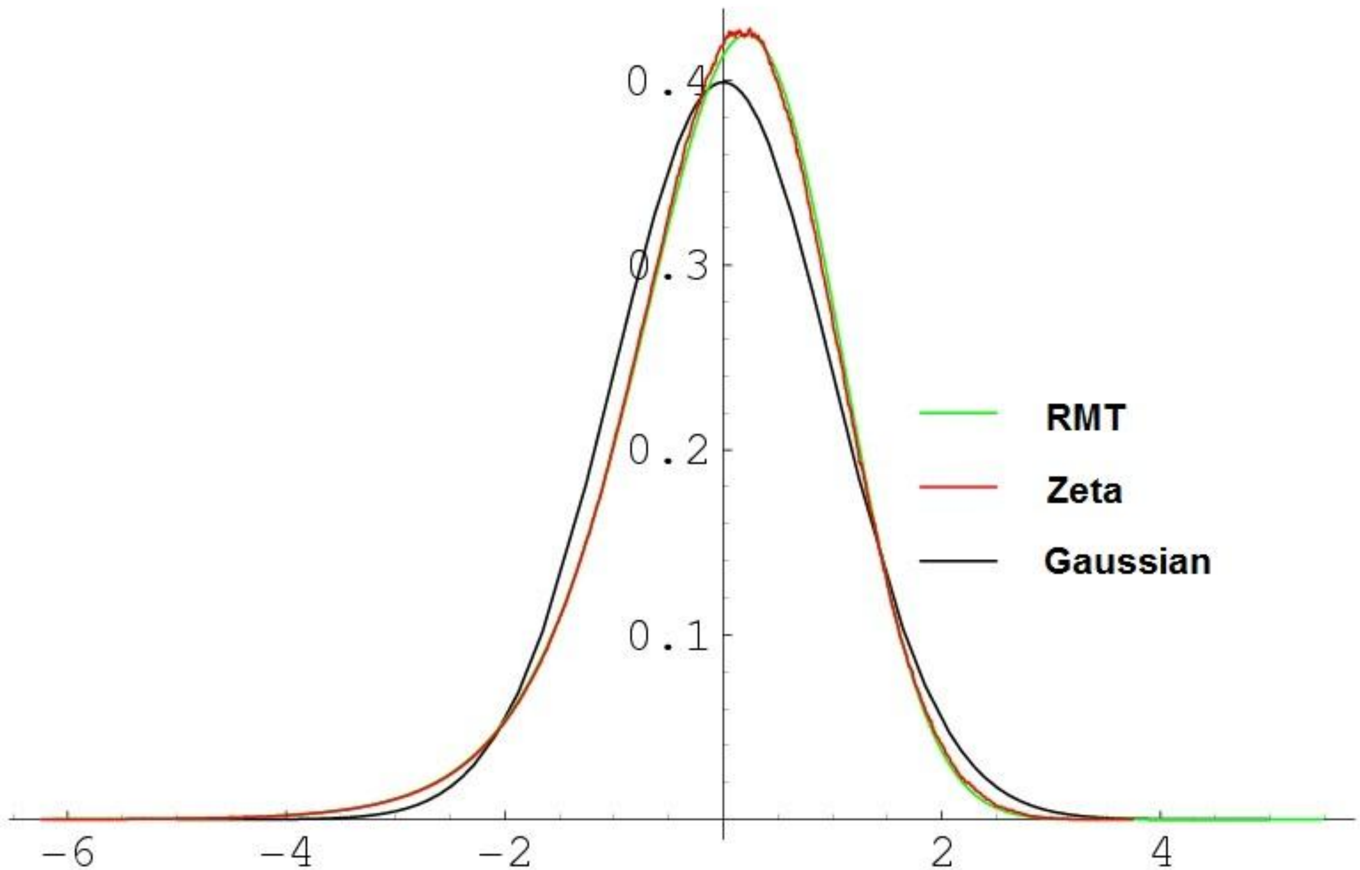
A serendipitous cup of tea



Nearest neighbour spacing



Random matrix model for zeta



Just a moment more…

k	$\frac{1}{T}\int_0^T \left\lvert \zeta\left(\frac{1}{2}+it\right)\right\rvert^{2k} dt$	
1	$\log(T)$	
2	$\frac{1}{12}\frac{6}{\pi^2}\log(T)^4$	
3	$\frac{42}{9!}a(3)\log(T)^9$	
4	$\frac{24024}{16!}a(4)\log(T)^{16}$	
5	?	

Just a moment more…

k	$\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt$	$\mathbb{E}\left[\lvert Z(U)\rvert^{2k}\right]$
1	$\log(T)$	N
2	$\frac{1}{12}\frac{6}{\pi^2}\log(T)^4$	$\frac{1}{12}N^4$
3	$\frac{42}{9!}a(3)\log(T)^9$	$\frac{42}{9!}N^9$
4	$\frac{24024}{16!}a(4)\log(T)^{16}$	$\frac{24024}{16!}N^{16}$
5	?	$\frac{G(k+1)^2}{G(2k+1)}N^{k^2}$

Keating-Snaith conjecture

$$\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \sim \frac{G(k+1)^2}{G(2k+1)} a(k) \log(T)^{k^2}$$

where

$$a(k) = \prod_{p\ prime} \left\{ \left(1-\frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \frac{\Gamma(m+k)^2}{(m!)^2 \Gamma(k)^2} p^{-m} \right\}$$

Quantum chaos

- How can one tell if a quantum system is classically chaotic?
- Answer: Look at the energy levels.
- Integrable = Poisson Chaotic = RMT

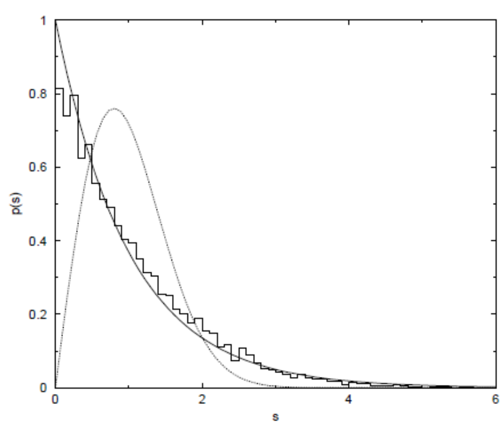


Fig. 18. The nearest neighbor distribution for 10000 first levels of the triangle $(\pi/2, \pi/3, 0)$ (the modular triangle). Solid line - the Poisson distribution. Dotted line - the GOE distribution.

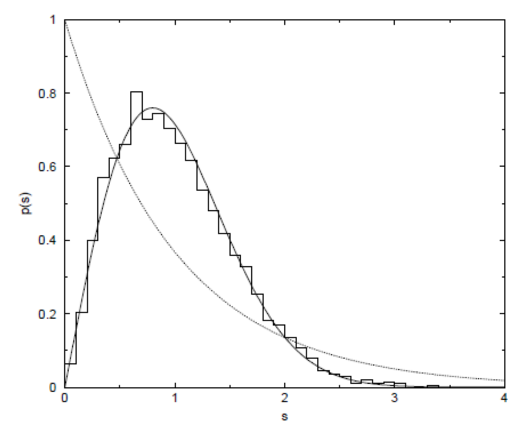


Fig. 19. The nearest-neighbor distribution of 6000 energy levels for the non-arithmetic Hecke triangular billiard with $n = 5$. The solid line – the GOE prediction. Dotted line – the Poisson result.

Further applications

- Bus arrival times in Cuernavaca (Mexico)
- Car parking in London
- Correlation matrix of time series of stock prices
- Sea-level and atmospheric pressure
- Longest increasing subsequence and Solitaire
- Brownian motion and non-intersecting random walks
- Disordered systems

Where to study?

- Bristol
- Brunel, London
- Kings College London
- Nottingham
- Queen Mary, London
- Warwick
- York